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OPERATIONS ON INTERNAL AND EXTERNAL CUBIC SOFT MATRICES

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Abstract:

In this paper, we define some operations on internal and external cubic soft matrices. We also characterized some of its algebraic properties.

Key Words: Cubic Soft Sets, Internal Cubic Soft Sets, External Cubic Soft Sets, Fuzzy Matrices & Internal and External Cubic Soft Matrices.

1. Introduction:

The concept of fuzzy set was introduced by Zadeh[12]. He studied their properties on the parallel lines to set theory. Since their many papers on fuzzy sets appeared high lighting the importance of the concept and its applications to logic set theory, group theory, real analysis, measure theory, topology etc.

In 1975, Zedeh[12] introduced a new notion of fuzzy sets viz., internal valued fuzzy subsets, where the values of the membership function are closed interval of numbers instead of a number.

In 1999, Molodstov [8] introduced the novel concept of softsets and established the fundamental results of the new theory. In [3], Jun *et al.* introduced a new notion of cubic set which is a combinations of fuzzy set and interval valued fuzzy set and also investigated several properties of cubic sets.

Fuzzy matrix was introduced by Thomason [11] and the concept of uncertainty was discussed by using fuzzy matrices. Chinnadurai and Barkavi introduced a new concept of cubic soft matrrix, interval cubic soft matrix and external cubic soft matrix[2].

In this paper, we define some operations on internal and external cubic soft matrix. We also discuss some basic algebraic properties of cubic soft matrix.

2. Preliminiaries:

In this section first we review some basic concepts and definitions.

Definition 2.1 [9] Let U be an initial universal set and E be a set of parameters. A cubic soft set over U is defined to be a pair (Φ, A) where Φ is a mapping from A to $P(U)$ and $A \subseteq E.$ Then the pair (Φ, A) can be represented as, $(\Phi, A) = \{\Phi(e)/e \in A\}$ where $\Phi(e) = \{\langle u, \widetilde{A}_e(u), \lambda_e(u)\rangle / u \in U, e \in A\}$ is a cubic soft set in which $\widetilde{A}_e(u)$ is the interval valued fuzzy set and $\lambda_e(u)$ is a fuzzy set.

Definition 2.2 [9] Let U be an initial universal set and E be a set of parameters. A cubic soft set (Φ, A) over U is said to be an internal cubic soft set if $A_e^-(u) \leq \lambda_e(u) \leq A_e^+(u)$ for all $e \in A$ and for all $u \in U.$

Definition 2.3 [9] Let U be an initial universal set and E be a set of parameters. A cubic soft set (Φ, A) over U is said to be an external cubic soft set if $\lambda_e(u) \notin (A_e^-(u), A_e^+(u))$ for all $e \in A$ and for all $u \in U.$

Definition 2.4 [9] Let U be an initial universal set and E be a set of parameters. For any subsets A and B of E, (Φ, A) and (Γ, B) be cubic soft sets over U.

1. The R-union of (Φ, A) and (Γ, B) is a cubic soft set (H, C) where $C = A \cup B$ and

$$\mathrm{H}e) = \begin{cases} \Phi(e) & \text{if } e \in A/B, \\ \Gamma(e) & \text{if } e \in B/A, \\ \Phi(e) \cup_R \Gamma(e) & \text{if} e \in A \cap B \end{cases}$$

for all $e \in C$. This is denoted by $(\mathrm{H}, C) = (\Phi, A) \cup_R (\Gamma, B).$

2. The R-intersection of (Φ, A) and (Γ, B) is a cubic soft set (H, C) where $C = A \cap B$ and

$$\mathrm{H}e) = \begin{cases} \Phi(e) & \text{if } e \in A/B, \\ \Gamma(e) & \text{if } e \in B/A, \\ \Phi(e) \cap_R \Gamma(e) & \text{if} e \in A \cap B \end{cases}$$

for all $e \in C$. This is denoted by $(\mathrm{H}, C) = (\Phi, A) \cap_R (\Gamma, B)$.

Definition 2.5 [9] Let U be an initial universal set and E be a set of parameters. For any subsets A and B of E, (Φ, A) and (Γ, B) be cubic soft sets over U.

1. The P-union of (Φ, A) and (Γ, B) is a cubic soft set (H, C) where $C = A \cup B$ and

$$\mathrm{H}e) = \begin{cases} \Phi(e) & \text{if } e \in A/B, \\ \Gamma(e) & \text{if } e \in B/A, \\ \Phi(e) \cup_P \Gamma(e) & \text{if} e \in A \cap B \end{cases}$$

for all $e \in C$. This is denoted by $(\mathrm{H}, C) = (\Phi, A) \cup_P (\Gamma, B)$.

2. The P-intersection of (Φ, A) and (Γ, B) is a cubic soft set (H, C) where $C = A \cap B$ and

$$\mathrm{H}e) = \begin{cases} \Phi(e) & \text{if } e \in A/B, \\ \Gamma(e) & \text{if } e \in B/A, \\ \Phi(e) \cap_P \Gamma(e) & \text{if} e \in A \cap B \end{cases}$$

for all $e \in C$. This is denoted by $(\mathrm{H}, C) = (\Phi, A) \cap_P (\Gamma, B)$.

Definition 2.6 [9] Let U be an initial universal set and E be a set of parameters. The complement of a cubic soft set (Φ, A) over U is denoted by $(\Phi, A)^c$ and defined by

$(\Phi, A) = (\Phi^c, \neg A)$ where $\Phi^c : \neg A \to XP(U)$ and $(\Phi, A)^c = \{\Phi^c(e)/e \in A\}$ where

$\Phi^c(e) = \{\langle u, \tilde{A}_e^c(u), \lambda_e^c(u)\rangle / u \in U, e \in A\}$

Definition 2.7 [7] A matrix $A = [a_{ij}]_{m \times n}$ is said to be fuzzy matrix if $a_{ij} \in [0,1], 1 \le i \le m$ and $1 \le j \le n$.

Definition 2.8 [7] For any two fuzzy matrices $A = [a_{ij}], B = [b_{ij}]$ and a scalar $k \in F$. Then,

(i) $A + B = [sup\{a_{ij}, b_{ij}\}] = \vee[a_{ij}, bij]$.

(ii) $AB = [sup\{inf\{a_{ij}, b_{ij}\}\}] = \vee\{\wedge[a_{ij}, b_{ij}]\}$.

(iii) $kA = [inf\{k, a_{ij}\}] = \wedge[k, a_{ij}]$.

Definition 2.9 [7] For any fuzzy matrix $A = [a_{ij}]$, the transpose is obtained by interchanging its rows and columns and is denoted by $A^T = [a_{ji}]$ for all i, j.

Definition 2.10 Let $U = \{u_1, u_2, \ldots, u_m\}$ be an initial universal set and $E = \{e_1, e_2, \ldots, e_n\}$ be a set of parameters. Let $A \subseteq E.$ Then cubic soft set (Φ, A) can be expressed in matrix form as

$$A^(= [a_{ij}] = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

such that $A^(= [a_{ij}] = \left\langle \tilde{A}_{e_j}(u_i), \lambda_{e_j}(u_i) \right\rangle = \left\langle \tilde{A}_{ij}^a, \lambda_{ij}^a \right\rangle$ which is called an $m \times n$ cubic soft matrix(shortly CS-matrix or CSM) of the cubic soft set (Φ, A).

According to this definition, a cubic soft set (Φ, A) is uniquely characterized by matrix $[a_{ij}]_{m \times n}$ where $i = 1,2,3,\ldots,m$ and $j = 1,2,3,\ldots,n.$

Example 2.11 Let $U = \{u_1, u_2, u_3, u_4\}$ is a set of cars and $A = \{e_1, e_2, e_3\}$ is a set of parameters, which stands for mileage, engine and prize respectively. Then cubic soft set is defined as

$$
\begin{aligned}
(\Phi, A) = \{ \quad & [e_1, (u_1, \langle[0.5,0.8],0.6\rangle), (u_2, \langle[0.1,0.7],0.5\rangle), (u_3, \langle[0.2,0.6],0.9\rangle), (u_4, \langle[0.3,0.9],0.4\rangle)], \\
& [e_2, (u_1, \langle[0.2,0.5],0.3\rangle), (u_2, \langle[0.3,0.6],0.7\rangle), (u_3, \langle[0.2,0.7],0.2\rangle), (u_4, \langle[0.3,0.5],0.1\rangle)], \\
& [e_3, (u_1, \langle[0.1,0.8],0.4\rangle), (u_2, \langle[0.6,0.7],0.9\rangle), (u_3, \langle[0.2,0.9],0.5\rangle), (u_4, \langle[0.3,0.7],0.4\rangle)]\}.
\end{aligned}
$$

Then the CS-matrix $\breve{A}$ is written as,

$$
\breve{A} = \begin{bmatrix}
\langle[0.5,0.8],0.6\rangle & \langle[0.2,0.5],0.3\rangle & \langle[0.1,0.8],0.4\rangle \\
\langle[0.1,0.7],0.5\rangle & \langle[0.3,0.6],0.7\rangle & \langle[0.6,0.7],0.9\rangle \\
\langle[0.2,0.6],0.7\rangle & \langle[0.2,0.7],0.2\rangle & \langle[0.2,0.9],0.5\rangle \\
\langle[0.3,0.9],0.4\rangle & \langle[0.3,0.5],0.1\rangle & \langle[0.3,0.7],0.4\rangle
\end{bmatrix}.
$$

3. Operations on Internal and External Cubic Soft Matrices:

In this section we define operations on internal and external cubic soft matrices and discuss some algebraic properties.

Definition 3.1 [2] Let $\breve{A} = [a_{ij}] \in CSM_{m\times n}$. Then $\breve{A}$ is an internal cubic soft matrix (ICSM), if $\tilde{A}_{ij}^{a^-} \le \lambda_{ij}^a \le \tilde{A}_{ij}^{a^+}$ for all i, j.

Definition 3.2 [2] Let $\breve{A} = [a_{ij}] \in CSM_{m\times n}$. Then $\breve{A}$ is an external cubic soft matrix (ECSM), if $\lambda_{ij}^a \notin (\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+})$ for all i, j.

Definition 3.3 [2] Let $\breve{A} = \left\langle [\tilde{A}_{ij}^{a^-}, \tilde{A}_{ij}^{a^+}], \lambda_{ij}^a \right\rangle, \breve{B} = \left\langle [\tilde{B}_{ij}^{b^-}, \tilde{B}_{ij}^{b^+}], \mu_{ij}^b \right\rangle \in CSM_{m\times n}$. Then

1. P-union of $\breve{A}$ and $\breve{B}$ is denoted by $\breve{A} \vee_P \breve{B}$ and defined as $\breve{A} \vee_P \breve{B} = \breve{C}$, if $\breve{C} = [c_{ij}] = \left\langle \tilde{C}_{ij}^c, \gamma_{ij}^c \right\rangle$, where $\tilde{C}_{ij}^c = \max\{\tilde{A}_{ij}^a, \tilde{B}_{ij}^b\}$ and $\gamma_{ij}^c = \max\{\lambda_{ij}^a, \mu_{ij}^b\}$ for all i, j.
2. P-intersection of $\breve{A}$ and $\breve{B}$ is denoted by $\breve{A} \wedge_P \breve{B}$ and defined as $\breve{A} \wedge_P \breve{B} = \breve{C}$, if $\breve{C} = [c_{ij}] = \left\langle \tilde{C}_{ij}^c, \gamma_{ij}^c \right\rangle$, where $\tilde{C}_{ij}^c = \min\{\tilde{A}_{ij}^a, \tilde{B}_{ij}^b\}$ and $\gamma_{ij}^c = \min\{\lambda_{ij}^a, \mu_{ij}^b\}$ for all i, j.
3. R-union of $\breve{A}$ and $\breve{B}$ is denoted by $\breve{A} \vee_R \breve{B}$ and defined as $\breve{A} \vee_R \breve{B} = \breve{C}$, if $\breve{C} = [c_{ij}] = \left\langle \tilde{C}_{ij}^c, \gamma_{ij}^c \right\rangle$, where $\tilde{C}_{ij}^c = \max\{\tilde{A}_{ij}^a, \tilde{B}_{ij}^b\}$ and $\gamma_{ij}^c = \min\{\lambda_{ij}^a, \mu_{ij}^b\}$ for all i, j.
4. R-intersection of $\breve{A}$ and $\breve{B}$ is denoted by $\breve{A} \wedge_R \breve{B}$ and defined as $\breve{A} \wedge_R \breve{B} = \breve{C}$, if $\breve{C} = [c_{ij}] = \left\langle \tilde{C}_{ij}^c, \gamma_{ij}^c \right\rangle$, where $\tilde{C}_{ij}^c = \min\{\tilde{A}_{ij}^a, \tilde{B}_{ij}^b\}$ and $\gamma_{ij}^c = \max\{\lambda_{ij}^a, \mu_{ij}^b\}$ for all i, j.

Proposition 3.4 *Let* $\breve{A}, \breve{B}, \breve{C} \in ICSM_{m\times n} (resp. ECSM_{m\times n})$. *Then*

$(i)\ \breve{A} \vee_P \breve{A} = \breve{A}$.

$(ii)\ \breve{A} \vee_R \breve{A} = \breve{A}$.

$(iii)\ \breve{A} \wedge_P \breve{A} = \breve{A}$.

$(iv)\ \breve{A} \wedge_R \breve{A} = \breve{A}$.

Proof. Let $\breve{A}, \breve{B}, \breve{C} \in ICSM_{m\times n}$.

(i) $\breve{A} \vee_P \breve{A} = \left\langle \tilde{A}_{ij}^a, \lambda_{ij}^a \right\rangle \vee_P \left\langle \tilde{A}_{ij}^a, \lambda_{ij}^a \right\rangle$ for all i, j.

$$
\begin{aligned}
\breve{A} \vee_P \breve{A} &= \left\langle \max\{\tilde{A}_{ij}^a, \tilde{A}_{ij}^a\}, \max\{\lambda_{ij}^a, \lambda_{ij}^a\} \right\rangle \text{ for all i, j} \\
&= \left\langle \tilde{A}_{ij}^a, \lambda_{ij}^a \right\rangle \\
&= \breve{A}.
\end{aligned}
$$

(ii) $\breve{A} \vee_R \breve{A} = \left\langle \tilde{A}_{ij}^a, \lambda_{ij}^a \right\rangle \vee_R \left\langle \tilde{A}_{ij}^a, \lambda_{ij}^a \right\rangle$ for all i, j.

$$\begin{aligned} \breve{A} \vee_R \breve{A} &= \langle \max\{\tilde{A}^a_{ij}, \tilde{A}^a_{ij}\}, \min\{\lambda^a_{ij}, \lambda^a_{ij}\} \rangle \text{ for all i, j} \\ &= \langle \tilde{A}^a_{ij}, \lambda^a_{ij} \rangle \\ &= \breve{A}. \end{aligned}$$

(iii) $\breve{A} \wedge_P \breve{A} = \langle \tilde{A}^a_{ij}, \lambda^a_{ij} \rangle \wedge_P \langle \tilde{A}^a_{ij}, \lambda^a_{ij} \rangle$ for all $i, j.$

$$\begin{aligned} \breve{A} \wedge_P \breve{A} &= \langle \min\{\tilde{A}^a_{ij}, \tilde{A}^a_{ij}\}, \min\{\lambda^a_{ij}, \lambda^a_{ij}\} \rangle \text{ for all i, j} \\ &= \langle \tilde{A}^a_{ij}, \lambda^a_{ij} \rangle \\ &= \breve{A}. \end{aligned}$$

(iv) $\breve{A} \wedge_R \breve{A} = \langle \tilde{A}^a_{ij}, \lambda^a_{ij} \rangle \wedge_R \langle \tilde{A}^a_{ij}, \lambda^a_{ij} \rangle$ for all $i, j.$

$$\begin{aligned} \breve{A} \wedge_R \breve{A} &= \langle \min\{\tilde{A}^a_{ij}, \tilde{A}^a_{ij}\}, \max\{\lambda^a_{ij}, \lambda^a_{ij}\} \rangle \text{ for all i, j} \\ &= \langle \tilde{A}^a_{ij}, \lambda^a_{ij} \rangle \\ &= \breve{A}. \end{aligned}$$

Proposition 3.5 *Let* $\breve{A}, \breve{B} \in ICSM_{m\times n}(resp. ECSM_{m\times n})$. *Then*

(i) $\breve{A} \vee_P \breve{B} = \breve{B} \vee_P \breve{A}$.

(ii) $\breve{A} \vee_R \breve{B} = \breve{B} \vee_R \breve{A}$.

(iii) $\breve{A} \wedge_P \breve{B} = \breve{B} \wedge_P \breve{A}$.

(iv) $\breve{A} \wedge_R \breve{B} = \breve{B} \wedge_R \breve{A}$.

Proof. (i) $\breve{A} \vee_P \breve{B} = \langle \tilde{A}^a_{ij}, \lambda^a_{ij} \rangle \vee_P \langle \tilde{B}^b_{ij}, \mu^b_{ij} \rangle$ for all $i, j.$

$$\begin{aligned} \breve{A} \vee_P \breve{B} &= \langle \max\{\tilde{A}^a_{ij}, \tilde{B}^b_{ij}\}, \max\{\lambda^a_{ij}, \mu^b_{ij}\} \rangle \text{ for all i, j} \\ &= \langle \max\{\tilde{B}^b_{ij}, \tilde{A}^a_{ij}\}, \max\{\mu^b_{ij}, \lambda^a_{ij}\} \rangle \text{ for all i, j} \\ &= \breve{B} \vee_P \breve{A}. \end{aligned}$$

(ii) $\breve{A} \vee_R \breve{B} = \langle \tilde{A}^a_{ij}, \lambda^a_{ij} \rangle \vee_R \langle \tilde{B}^b_{ij}, \mu^b_{ij} \rangle$ for all $i, j.$

$$\begin{aligned} \breve{A} \vee_R \breve{B} &= \langle \max\{\tilde{A}^a_{ij}, \tilde{B}^b_{ij}\}, \min\{\lambda^a_{ij}, \mu^b_{ij}\} \rangle \text{ for all i, j} \\ &= \langle \max\{\tilde{B}^b_{ij}, \tilde{A}^a_{ij}\}, \min\{\mu^b_{ij}, \lambda^a_{ij}\} \rangle \text{ for all i, j} \\ &= \breve{B} \vee_R \breve{A}. \end{aligned}$$

(iii) $\breve{A} \wedge_P \breve{B} = \langle \tilde{A}^a_{ij}, \lambda^a_{ij} \rangle \wedge_P \langle \tilde{B}^b_{ij}, \mu^b_{ij} \rangle$ for all $i, j.$

$$\begin{aligned} \breve{A} \wedge_P \breve{B} &= \langle \min\{\tilde{A}^a_{ij}, \tilde{B}^b_{ij}\}, \min\{\lambda^a_{ij}, \mu^b_{ij}\} \rangle \text{ for all i, j} \\ &= \langle \min\{\tilde{B}^b_{ij}, \tilde{A}^a_{ij}\}, \min\{\mu^b_{ij}, \lambda^a_{ij}\} \rangle \text{ for all i, j} \\ &= \breve{B} \wedge_P \breve{A}. \end{aligned}$$

(iv) $\breve{A} \wedge_R \breve{B} = \langle \tilde{A}^a_{ij}, \lambda^a_{ij} \rangle \wedge_R \langle \tilde{B}^b_{ij}, \mu^b_{ij} \rangle$ for all $i, j.$

$$\begin{aligned} \breve{A} \wedge_R \breve{B} &= \langle \min\{\tilde{A}^a_{ij}, \tilde{B}^b_{ij}\}, \max\{\lambda^a_{ij}, \mu^b_{ij}\} \rangle \text{ for all i, j} \\ &= \langle \min\{\tilde{B}^b_{ij}, \tilde{A}^a_{ij}\}, \max\{\mu^b_{ij}, \lambda^a_{ij}\} \rangle \text{ for all i, j} \\ &= \breve{B} \wedge_R \breve{A}. \end{aligned}$$

Note 3.6 In Proposition 3.5, (i) and (iii) are also an ICSM. But (ii) and (iv) are not an ICSM.

Proposition 3.7 *Let* $\breve{A}, \breve{B}, \breve{C} \in ICSM_{m\times n}(resp. ECSM_{m\times n})$. *Then*

(i) $\left(\breve{A} \vee_P \breve{B}\right) \vee_P \breve{C} = \breve{A} \vee_P \left(\breve{B} \vee_P \breve{C}\right)$

(ii) $\left(\breve{A} \vee_R \breve{B}\right) \vee_R \breve{C} = \breve{A} \vee_R \left(\breve{B} \vee_R \breve{C}\right)$

(iii) $\left(\breve{A} \wedge_P \breve{B}\right) \wedge_P \breve{C} = \breve{A} \wedge_P \left(\breve{B} \wedge_P \breve{C}\right)$

(iv) $\left(\breve{A} \wedge_R \breve{B}\right) \wedge_R \breve{C} = \breve{A} \wedge_R \left(\breve{B} \wedge_R \breve{C}\right)$

Proof. Let $\breve{A} = \left\langle \tilde{A}_{ij}^a, \lambda_{ij}^a \right\rangle, \breve{B} = \left\langle \tilde{B}_{ij}^b, \mu_{ij}^b \right\rangle, \breve{C} = \left\langle \tilde{C}_{ij}^c, \gamma_{ij}^c \right\rangle \in ICSM_{m\times n}$.

Then (i) $\left(\breve{A} \vee_P \breve{B}\right) \vee_P \breve{C} = \left(\left\langle \tilde{A}_{ij}^a, \lambda_{ij}^a \right\rangle \vee_P \left\langle \tilde{B}_{ij}^b, \mu_{ij}^b \right\rangle\right) \vee_P \left\langle \tilde{C}_{ij}^c, \gamma_{ij}^c \right\rangle$.

$$\begin{aligned}
\left(\breve{A} \vee_P \breve{B}\right) \vee_R \breve{C} &= \left(\left\langle \max\{\tilde{A}_{ij}^a, \tilde{B}_{ij}^b\}, \max\{\lambda_{ij}^a, \mu_{ij}^b\} \right\rangle\right) \vee_P \breve{C} \text{ for all i, j} \\
&= \left\langle \max\{\max\{\tilde{A}_{ij}^a, \tilde{B}_{ij}^b\}, \tilde{C}_{ij}^c\}, \max\{\max\{\lambda_{ij}^a, \mu_{ij}^b\}, \gamma_{ij}^c\} \right\rangle \\
&= \left\langle \max\{\tilde{A}_{ij}^a, \max\{\tilde{B}_{ij}^b, \tilde{C}_{ij}^c\}\}, \max\{\lambda_{ij}^a, \max\{\mu_{ij}^b, \gamma_{ij}^c\}\} \right\rangle \\
&= \breve{A} \vee_P \left(\left\langle \max\{\tilde{B}_{ij}^b, \tilde{C}_{ij}^c\}, \max\{\mu_{ij}^b, \gamma_{ij}^c\} \right\rangle\right) \text{for all i, j} \\
&= \breve{A} \vee_P \left(\breve{B} \vee_R \breve{C}\right)
\end{aligned}$$

(ii) $\left(\breve{A} \vee_R \breve{B}\right) \vee_R \breve{C} = \left(\left\langle \tilde{A}_{ij}^a, \lambda_{ij}^a \right\rangle \vee_R \left\langle \tilde{B}_{ij}^b, \mu_{ij}^b \right\rangle\right) \vee_R \left\langle \tilde{C}_{ij}^c, \gamma_{ij}^c \right\rangle$.

$$\begin{aligned}
\left(\breve{A} \vee_R \breve{B}\right) \vee_R \breve{C} &= \left(\left\langle \max\{\tilde{A}_{ij}^a, \tilde{B}_{ij}^b\}, \min\{\lambda_{ij}^a, \mu_{ij}^b\} \right\rangle\right) \vee_R \breve{C} \text{ for all i, j} \\
&= \left\langle \max\{\max\{\tilde{A}_{ij}^a, \tilde{B}_{ij}^b\}, \tilde{C}_{ij}^c\}, \min\{\min\{\lambda_{ij}^a, \mu_{ij}^b\}, \gamma_{ij}^c\} \right\rangle \\
&= \left\langle \max\{\tilde{A}_{ij}^a, \max\{\tilde{B}_{ij}^b, \tilde{C}_{ij}^c\}\}, \min\{\lambda_{ij}^a, \min\{\mu_{ij}^b, \gamma_{ij}^c\}\} \right\rangle \\
&= \breve{A} \vee_R \left(\left\langle \max\{\tilde{B}_{ij}^b, \tilde{C}_{ij}^c\}, \min\{\mu_{ij}^b, \gamma_{ij}^c\} \right\rangle\right) \text{for all i, j} \\
&= \breve{A} \vee_R \left(\breve{B} \vee_R \breve{C}\right)
\end{aligned}$$

(iii) $\left(\breve{A} \wedge_P \breve{B}\right) \wedge_P \breve{C} = \left(\left\langle \tilde{A}_{ij}^a, \lambda_{ij}^a \right\rangle \wedge_P \left\langle \tilde{B}_{ij}^b, \mu_{ij}^b \right\rangle\right) \wedge_P \left\langle \tilde{C}_{ij}^c, \gamma_{ij}^c \right\rangle$.

$$\begin{aligned}
\left(\breve{A} \wedge_P \breve{B}\right) \wedge_P \breve{C} &= \left(\left\langle \min\{\tilde{A}_{ij}^a, \tilde{B}_{ij}^b\}, \min\{\lambda_{ij}^a, \mu_{ij}^b\} \right\rangle\right) \vee_P \breve{C} \text{ for all i, j} \\
&= \left\langle \min\{\min\{\tilde{A}_{ij}^a, \tilde{B}_{ij}^b\}, \tilde{C}_{ij}^c\}, \min\{\min\{\lambda_{ij}^a, \mu_{ij}^b\}, \gamma_{ij}^c\} \right\rangle \\
&= \left\langle \min\{\tilde{A}_{ij}^a, \min\{\tilde{B}_{ij}^b, \tilde{C}_{ij}^c\}\}, \min\{\lambda_{ij}^a, \min\{\mu_{ij}^b, \gamma_{ij}^c\}\} \right\rangle \\
&= \breve{A} \wedge_P \left(\left\langle \min\{\tilde{B}_{ij}^b, \tilde{C}_{ij}^c\}, \min\{\mu_{ij}^b, \gamma_{ij}^c\} \right\rangle\right) \text{for all i, j} \\
&= \breve{A} \wedge_P \left(\breve{B} \wedge_P \breve{C}\right)
\end{aligned}$$

(iv) $\left(\breve{A} \wedge_R \breve{B}\right) \wedge_R \breve{C} = \left(\left\langle \tilde{A}_{ij}^a, \lambda_{ij}^a \right\rangle \wedge_R \left\langle \tilde{B}_{ij}^b, \mu_{ij}^b \right\rangle\right) \wedge_R \left\langle \tilde{C}_{ij}^c, \gamma_{ij}^c \right\rangle$.

$$\begin{aligned}
\left(\breve{A} \wedge_R \breve{B}\right) \wedge_R \breve{C} &= \left(\left\langle \min\{\tilde{A}_{ij}^a, \tilde{B}_{ij}^b\}, \max\{\lambda_{ij}^a, \mu_{ij}^b\} \right\rangle\right) \vee_R \breve{C} \text{ for all i, j} \\
&= \left\langle \min\{\min\{\tilde{A}_{ij}^a, \tilde{B}_{ij}^b\}, \tilde{C}_{ij}^c\}, \max\{\max\{\lambda_{ij}^a, \mu_{ij}^b\}, \gamma_{ij}^c\} \right\rangle \\
&= \left\langle \min\{\tilde{A}_{ij}^a, \min\{\tilde{B}_{ij}^b, \tilde{C}_{ij}^c\}\}, \max\{\lambda_{ij}^a, \max\{\mu_{ij}^b, \gamma_{ij}^c\}\} \right\rangle \\
&= \breve{A} \wedge_R \left(\left\langle \min\{\tilde{B}_{ij}^b, \tilde{C}_{ij}^c\}, \max\{\mu_{ij}^b, \gamma_{ij}^c\} \right\rangle\right) \text{for all i, j} \\
&= \breve{A} \wedge_R \left(\breve{B} \wedge_R \breve{C}\right)
\end{aligned}$$

Note 3.8 In Proposition 3.7, (i) and (iii) are also an ICSM. But (ii) and (iv) are not an ICSM.

Proposition 3.9 *Let* $\breve{A}, \breve{B} \in ICSM_{m\times n}(resp. ECSM_{m\times n})$. *Then*

(i) $\left(\breve{A} \vee_P \breve{B}\right)^c = \left(\breve{A}\right)^c \wedge_P \left(\breve{B}\right)^c$.

(ii) $\left(\breve{A} \vee_R \breve{B}\right)^c = \left(\breve{A}\right)^c \wedge_R \left(\breve{B}\right)^c$.

(iii) $\left(\breve{A} \wedge_P \breve{B}\right)^c = \left(\breve{A}\right)^c \vee_P \left(\breve{B}\right)^c$.

(iv) $\left(\breve{A} \wedge_R \breve{B}\right)^c = \left(\breve{A}\right)^c \vee_R \left(\breve{B}\right)^c$.

Proof. Let $\breve{A} = \left\langle \tilde{A}^a_{ij}, \lambda^a_{ij} \right\rangle, \breve{B} = \left\langle \tilde{B}^b_{ij}, \mu^b_{ij} \right\rangle \in ICSM_{m\times n}$.

Then (i) $(\breve{A} \vee_P \breve{B})^c = \left(\left\langle \tilde{A}^a_{ij}, \lambda^a_{ij} \right\rangle \vee_P \left\langle \tilde{A}^a_{ij}, \lambda^a_{ij} \right\rangle\right)^c$ for all i, j.

$$
\begin{aligned}
(\breve{A} \vee_P \breve{B})^c &= 1-(\breve{A} \vee_P \breve{B}) \\
&= 1-\left\langle \max\{\tilde{A}^a_{ij}, \tilde{B}^b_{ij}\}, \max\{\lambda^a_{ij}, \mu^b_{ij}\} \right\rangle \text{ for all i, j} \\
&= \left\langle \min\{1-\tilde{A}^a_{ij}, 1-\tilde{B}^b_{ij}\}, \min\{1-\lambda^a_{ij}, 1-\mu^b_{ij}\} \right\rangle \\
&= \left\langle \min\{(\tilde{A}^a_{ij})^c, (\tilde{B}^b_{ij})^c\}, \min\{(\lambda^a_{ij})^c, (\mu^b_{ij})^c\} \right\rangle \\
&= \left(\breve{A}\right)^c \wedge_P \left(\breve{B}\right)^c.
\end{aligned}
$$

(ii) $(\breve{A} \vee_R \breve{B})^c = \left(\left\langle \tilde{A}^a_{ij}, \lambda^a_{ij} \right\rangle \vee_R \left\langle \tilde{A}^a_{ij}, \lambda^a_{ij} \right\rangle\right)^c$ for all i, j.

$$
\begin{aligned}
(\breve{A} \vee_R \breve{B})^c &= 1-(\breve{A} \vee_R \breve{B}) \\
&= 1-\left\langle \max\{\tilde{A}^a_{ij}, \tilde{B}^b_{ij}\}, \min\{\lambda^a_{ij}, \mu^b_{ij}\} \right\rangle \text{ for all i, j} \\
&= \left\langle \min\{1-\tilde{A}^a_{ij}, 1-\tilde{B}^b_{ij}\}, \max\{1-\lambda^a_{ij}, 1-\mu^b_{ij}\} \right\rangle \\
&= \left\langle \min\{(\tilde{A}^a_{ij})^c, (\tilde{B}^b_{ij})^c\}, \max\{(\lambda^a_{ij})^c, (\mu^b_{ij})^c\} \right\rangle \\
&= \left(\breve{A}\right)^c \wedge_R \left(\breve{B}\right)^c.
\end{aligned}
$$

(iii) $(\breve{A} \wedge_P \breve{B})^c = \left(\left\langle \tilde{A}^a_{ij}, \lambda^a_{ij} \right\rangle \wedge_P \left\langle \tilde{A}^a_{ij}, \lambda^a_{ij} \right\rangle\right)^c$ for all i, j.

$$
\begin{aligned}
(\breve{A} \wedge_P \breve{B})^c &= 1-(\breve{A} \wedge_P \breve{B}) \\
&= 1-\left\langle \min\{\tilde{A}^a_{ij}, \tilde{B}^b_{ij}\}, \min\{\lambda^a_{ij}, \mu^b_{ij}\} \right\rangle \text{ for all i, j} \\
&= \left\langle \max\{1-\tilde{A}^a_{ij}, 1-\tilde{B}^b_{ij}\}, \max\{1-\lambda^a_{ij}, 1-\mu^b_{ij}\} \right\rangle \\
&= \left\langle \max\{(\tilde{A}^a_{ij})^c, (\tilde{B}^b_{ij})^c\}, \max\{(\lambda^a_{ij})^c, (\mu^b_{ij})^c\} \right\rangle \\
&= \left(\breve{A}\right)^c \vee_P \left(\breve{B}\right)^c.
\end{aligned}
$$

(iv) $(\breve{A} \wedge_R \breve{B})^c = \left(\left\langle \tilde{A}^a_{ij}, \lambda^a_{ij} \right\rangle \wedge_R \left\langle \tilde{A}^a_{ij}, \lambda^a_{ij} \right\rangle\right)^c$ for all i, j.

$$
\begin{aligned}
(\breve{A} \wedge_R \breve{B})^c &= 1-(\breve{A} \wedge_R \breve{B}) \\
&= 1-\left\langle \min\{\tilde{A}^a_{ij}, \tilde{B}^b_{ij}\}, \max\{\lambda^a_{ij}, \mu^b_{ij}\} \right\rangle \text{ for all i, j} \\
&= \left\langle \max\{1-\tilde{A}^a_{ij}, 1-\tilde{B}^b_{ij}\}, \min\{1-\lambda^a_{ij}, 1-\mu^b_{ij}\} \right\rangle \\
&= \left\langle \max\{(\tilde{A}^a_{ij})^c, (\tilde{B}^b_{ij})^c\}, \min\{(\lambda^a_{ij})^c, (\mu^b_{ij})^c\} \right\rangle \\
&= \left(\breve{A}\right)^c \vee_R \left(\breve{B}\right)^c.
\end{aligned}
$$

Note 3.10 In Proposition 3.9, (i) and (iii) are also an ICSM. But (ii) and (iv) are not an ICSM.

Theorem 3.11 *Let* $\breve{A}, \breve{B}, \breve{C} \in ICSM_{m\times n}(resp.ECSM_{m\times n})$. *Then*

(i) $\left(\breve{A} \vee_P \breve{B}\right) \wedge_P \breve{C} = \left(\breve{A} \wedge_P \breve{C}\right) \vee_P \left(\breve{B} \wedge_P \breve{C}\right)$

(ii) $\left(\breve{A} \vee_P \breve{B}\right) \wedge_R \breve{C} = \left(\breve{A} \wedge_R \breve{C}\right) \vee_P \left(\breve{B} \wedge_R \breve{C}\right)$

(iii) $\left(\breve{A} \vee_R \breve{B}\right) \wedge_P \breve{C} = \left(\breve{A} \wedge_P \breve{C}\right) \vee_R \left(\breve{B} \wedge_P \breve{C}\right)$

(iv) $\left(\breve{A} \vee_R \breve{B}\right) \wedge_R \breve{C} = \left(\breve{A} \wedge_R \breve{C}\right) \vee_R \left(\breve{B} \wedge_R \breve{C}\right)$

(v) $\left(\breve{A} \wedge_P \breve{B}\right) \vee_P \breve{C} = \left(\breve{A} \vee_P \breve{C}\right) \wedge_P \left(\breve{B} \vee_P \breve{C}\right)$

(vi) $(\breve{A} \wedge_P \breve{B}) \vee_R \breve{C} = (\breve{A} \vee_R \breve{C}) \wedge_P (\breve{B} \vee_R \breve{C})$

(vii) $(\breve{A} \wedge_R \breve{B}) \vee_P \breve{C} = (\breve{A} \vee_P \breve{C}) \wedge_R (\breve{B} \vee_P \breve{C})$

(viii) $(\breve{A} \wedge_R \breve{B}) \vee_R \breve{C} = (\breve{A} \vee_R \breve{C}) \wedge_R (\breve{B} \vee_R \breve{C})$

Proof. Let $\breve{A} = \langle \tilde{A}_{ij}^a, \lambda_{ij}^a \rangle, \breve{B} = \langle \tilde{B}_{ij}^b, \mu_{ij}^b \rangle, \breve{C} = \langle \tilde{C}_{ij}^c, \gamma_{ij}^c \rangle \in ICSM_{m\times n}$.

Then (i) $(\breve{A} \vee_P \breve{B}) \wedge_P \breve{C} = (\langle \tilde{A}_{ij}^a, \lambda_{ij}^a \rangle \vee_P \langle \tilde{B}_{ij}^b \mu_{ij}^b \rangle) \wedge_P \langle \tilde{C}_{ij}^c, \gamma_{ij}^c \rangle$.

$$\begin{aligned}
(\breve{A} \vee_P \breve{B}) \wedge_P \breve{C} &= \langle \max\{\tilde{A}_{ij}^a, \tilde{B}_{ij}^b\}, \max\{\lambda_{ij}^a, \mu_{ij}^b\} \rangle \wedge_P \breve{C} \text{ for all i, j} \\
&= \langle \min\{\max\{\tilde{A}_{ij}^a, \tilde{B}_{ij}^b\}, \tilde{C}_{ij}^c\}, \min\{\max\{\lambda_{ij}^a, \mu_{ij}^b\}, \gamma_{ij}^c\} \rangle \\
&= \langle \max\{\min\{\tilde{A}_{ij}^a, \tilde{C}_{ij}^c\}, \min\{\tilde{B}_{ij}^b, \tilde{C}_{ij}^c\}\}, \max\{\min\{\lambda_{ij}^a, \gamma_{ij}^c\}, \min\{\mu_{ij}^b, \gamma_{ij}^c\}\} \rangle \\
&= (\breve{A} \wedge_P \breve{C}) \vee_P (\breve{B} \wedge_P \breve{C})
\end{aligned}$$

(ii) $(\breve{A} \vee_P \breve{B}) \wedge_R \breve{C} = (\langle \tilde{A}_{ij}^a, \lambda_{ij}^a \rangle \vee_P \langle \tilde{B}_{ij}^b \mu_{ij}^b \rangle) \wedge_R \langle \tilde{C}_{ij}^c, \gamma_{ij}^c \rangle$.

$$\begin{aligned}
(\breve{A} \vee_P \breve{B}) \wedge_R \breve{C} &= \langle \max\{\tilde{A}_{ij}^a, \tilde{B}_{ij}^b\}, \max\{\lambda_{ij}^a, \mu_{ij}^b\} \rangle \wedge_R \breve{C} \text{ for all i, j} \\
&= \langle \min\{\max\{\tilde{A}_{ij}^a, \tilde{B}_{ij}^b\}, \tilde{C}_{ij}^c\}, \max\{\max\{\lambda_{ij}^a, \mu_{ij}^b\}, \gamma_{ij}^c\} \rangle \\
&= \langle \max\{\min\{\tilde{A}_{ij}^a, \tilde{C}_{ij}^c\}, \min\{\tilde{B}_{ij}^b, \tilde{C}_{ij}^c\}\}, \max\{\max\{\lambda_{ij}^a, \gamma_{ij}^c\}, \max\{\mu_{ij}^b, \gamma_{ij}^c\}\} \rangle \\
&= (\breve{A} \wedge_R \breve{C}) \vee_P (\breve{B} \wedge_R \breve{C})
\end{aligned}$$

(iii) $(\breve{A} \vee_R \breve{B}) \wedge_P \breve{C} = (\langle \tilde{A}_{ij}^a, \lambda_{ij}^a \rangle \vee_R \langle \tilde{B}_{ij}^b \mu_{ij}^b \rangle) \wedge_P \langle \tilde{C}_{ij}^c, \gamma_{ij}^c \rangle$.

$$\begin{aligned}
(\breve{A} \vee_R \breve{B}) \wedge_P \breve{C} &= \langle \max\{\tilde{A}_{ij}^a, \tilde{B}_{ij}^b\}, \min\{\lambda_{ij}^a, \mu_{ij}^b\} \rangle \wedge_P \breve{C} \text{ for all i, j} \\
&= \langle \min\{\max\{\tilde{A}_{ij}^a, \tilde{B}_{ij}^b\}, \tilde{C}_{ij}^c\}, \min\{\min\{\lambda_{ij}^a, \mu_{ij}^b\}, \gamma_{ij}^c\} \rangle \\
&= \langle \max\{\min\{\tilde{A}_{ij}^a, \tilde{C}_{ij}^c\}, \min\{\tilde{B}_{ij}^b, \tilde{C}_{ij}^c\}\}, \min\{\min\{\lambda_{ij}^a, \gamma_{ij}^c\}, \min\{\mu_{ij}^b, \gamma_{ij}^c\}\} \rangle \\
&= (\breve{A} \wedge_P \breve{C}) \vee_R (\breve{B} \wedge_P \breve{C})
\end{aligned}$$

(iv) $(\breve{A} \vee_R \breve{B}) \wedge_R \breve{C} = (\langle \tilde{A}_{ij}^a, \lambda_{ij}^a \rangle \vee_R \langle \tilde{B}_{ij}^b \mu_{ij}^b \rangle) \wedge_R \langle \tilde{C}_{ij}^c, \gamma_{ij}^c \rangle$.

$$\begin{aligned}
(\breve{A} \vee_R \breve{B}) \wedge_R \breve{C} &= \langle \max\{\tilde{A}_{ij}^a, \tilde{B}_{ij}^b\}, \min\{\lambda_{ij}^a, \mu_{ij}^b\} \rangle \wedge_R \breve{C} \text{ for all i, j} \\
&= \langle \min\{\max\{\tilde{A}_{ij}^a, \tilde{B}_{ij}^b\}, \tilde{C}_{ij}^c\}, \max\{\min\{\lambda_{ij}^a, \mu_{ij}^b\}, \gamma_{ij}^c\} \rangle \\
&= \langle \max\{\min\{\tilde{A}_{ij}^a, \tilde{C}_{ij}^c\}, \min\{\tilde{B}_{ij}^b, \tilde{C}_{ij}^c\}\}, \min\{\max\{\lambda_{ij}^a, \gamma_{ij}^c\}, \max\{\mu_{ij}^b, \gamma_{ij}^c\}\} \rangle \\
&= (\breve{A} \wedge_R \breve{C}) \vee_R (\breve{B} \wedge_R \breve{C})
\end{aligned}$$

(v) $(\breve{A} \wedge_P \breve{B}) \vee_P \breve{C} = (\langle \tilde{A}_{ij}^a, \lambda_{ij}^a \rangle \wedge_P \langle \tilde{B}_{ij}^b \mu_{ij}^b \rangle) \vee_P \langle \tilde{C}_{ij}^c, \gamma_{ij}^c \rangle$.

$$\begin{aligned}
(\breve{A} \wedge_P \breve{B}) \vee_P \breve{C} &= \langle \min\{\tilde{A}_{ij}^a, \tilde{B}_{ij}^b\}, \min\{\lambda_{ij}^a, \mu_{ij}^b\} \rangle \vee_P \breve{C} \text{ for all i, j} \\
&= \langle \max\{\min\{\tilde{A}_{ij}^a, \tilde{B}_{ij}^b\}, \tilde{C}_{ij}^c\}, \max\{\min\{\lambda_{ij}^a, \mu_{ij}^b\}, \gamma_{ij}^c\} \rangle \\
&= \langle \min\{\max\{\tilde{A}_{ij}^a, \tilde{C}_{ij}^c\}, \max\{\tilde{B}_{ij}^b, \tilde{C}_{ij}^c\}\}, \min\{\max\{\lambda_{ij}^a, \gamma_{ij}^c\}, \max\{\mu_{ij}^b, \gamma_{ij}^c\}\} \rangle \\
&= (\breve{A} \vee_P \breve{C}) \wedge_P (\breve{B} \vee_P \breve{C})
\end{aligned}$$

(vi) $(\breve{A} \wedge_P \breve{B}) \vee_R \breve{C} = (\langle \tilde{A}_{ij}^a, \lambda_{ij}^a \rangle \wedge_P \langle \tilde{B}_{ij}^b \mu_{ij}^b \rangle) \vee_R \langle \tilde{C}_{ij}^c, \gamma_{ij}^c \rangle$.

$$\begin{aligned}
(\breve{A} \wedge_P \breve{B}) \vee_R \breve{C} &= \langle \min\{\tilde{A}_{ij}^a, \tilde{B}_{ij}^b\}, \min\{\lambda_{ij}^a, \mu_{ij}^b\} \rangle \vee_R \breve{C} \text{ for all i, j} \\
&= \langle \max\{\min\{\tilde{A}_{ij}^a, \tilde{B}_{ij}^b\}, \tilde{C}_{ij}^c\}, \min\{\min\{\lambda_{ij}^a, \mu_{ij}^b\}, \gamma_{ij}^c\} \rangle \\
&= \langle \min\{\max\{\tilde{A}_{ij}^a, \tilde{C}_{ij}^c\}, \max\{\tilde{B}_{ij}^b, \tilde{C}_{ij}^c\}\}, \min\{\min\{\lambda_{ij}^a, \gamma_{ij}^c\}, \min\{\mu_{ij}^b, \gamma_{ij}^c\}\} \rangle \\
&= (\breve{A} \vee_P \breve{C}) \wedge_R (\breve{B} \vee_P \breve{C})
\end{aligned}$$

(vii) $\left(\breve{A} \wedge_R \breve{B}\right) \vee_P \breve{C} = \left(\left\langle \tilde{A}_{ij}^{a}, \lambda_{ij}^{a} \right\rangle \wedge_R \left\langle \tilde{B}_{ij}^{b} \mu_{ij}^{b} \right\rangle\right) \vee_P \left\langle \tilde{C}_{ij}^{c}, \gamma_{ij}^{c} \right\rangle.$

$$\begin{aligned}
\left(\breve{A} \wedge_R \breve{B}\right) \vee_P \breve{C} &= \left\langle \min\{\tilde{A}_{ij}^{a}, \tilde{B}_{ij}^{b}\}, \max\{\lambda_{ij}^{a}, \mu_{ij}^{b}\} \right\rangle \vee_P \breve{C} \text{ for all i, j} \\
&= \left\langle \max\{\min\{\tilde{A}_{ij}^{a}, \tilde{B}_{ij}^{b}\}, \tilde{C}_{ij}^{c}\}, \max\{\max\{\lambda_{ij}^{a}, \mu_{ij}^{b}\}, \gamma_{ij}^{c}\} \right\rangle \\
&= \left\langle \min\{\max\{\tilde{A}_{ij}^{a}, \tilde{C}_{ij}^{c}\}, \max\{\tilde{B}_{ij}^{b}, \tilde{C}_{ij}^{c}\}\}, \max\{\max\{\lambda_{ij}^{a}, \gamma_{ij}^{c}\}, \max\{\mu_{ij}^{b}, \gamma_{ij}^{c}\}\} \right\rangle \\
&= \left(\breve{A} \vee_P \breve{C}\right) \wedge_R \left(\breve{B} \vee_P \breve{C}\right)
\end{aligned}$$

(viii) $\left(\breve{A} \wedge_R \breve{B}\right) \vee_R \breve{C} = \left(\left\langle \tilde{A}_{ij}^{a}, \lambda_{ij}^{a} \right\rangle \wedge_R \left\langle \tilde{B}_{ij}^{b} \mu_{ij}^{b} \right\rangle\right) \vee_R \left\langle \tilde{C}_{ij}^{c}, \gamma_{ij}^{c} \right\rangle.$

$$\begin{aligned}
\left(\breve{A} \wedge_R \breve{B}\right) \vee_R \breve{C} &= \left\langle \min\{\tilde{A}_{ij}^{a}, \tilde{B}_{ij}^{b}\}, \max\{\lambda_{ij}^{a}, \mu_{ij}^{b}\} \right\rangle \vee_R \breve{C} \text{ for all i, j} \\
&= \left\langle \max\{\min\{\tilde{A}_{ij}^{a}, \tilde{B}_{ij}^{b}\}, \tilde{C}_{ij}^{c}\}, \min\{\max\{\lambda_{ij}^{a}, \mu_{ij}^{b}\}, \gamma_{ij}^{c}\} \right\rangle \\
&= \left\langle \min\{\max\{\tilde{A}_{ij}^{a}, \tilde{C}_{ij}^{c}\}, \max\{\tilde{B}_{ij}^{b}, \tilde{C}_{ij}^{c}\}\}, \max\{\min\{\lambda_{ij}^{a}, \gamma_{ij}^{c}\}, \min\{\mu_{ij}^{b}, \gamma_{ij}^{c}\}\} \right\rangle \\
&= \left(\breve{A} \vee_R \breve{C}\right) \wedge_R \left(\breve{B} \vee_R \breve{C}\right)
\end{aligned}$$

Theorem 3.12 *Let* $\breve{A}, \breve{B}, \breve{C} \in ICSM_{m\times n}(resp.ECSM_{m\times n})$. *Then*

(i) $\breve{A} \vee_P \left(\breve{B} \wedge_P \breve{C}\right) = \left(\breve{A} \vee_P \breve{B}\right) \wedge_P \left(\breve{A} \vee_P \breve{C}\right)$

(ii) $\breve{A} \vee_P \left(\breve{B} \wedge_R \breve{C}\right) = \left(\breve{A} \vee_P \breve{B}\right) \wedge_R \left(\breve{A} \vee_P \breve{C}\right)$

(iii) $\breve{A} \vee_R \left(\breve{B} \wedge_P \breve{C}\right) = \left(\breve{A} \vee_R \breve{B}\right) \wedge_P \left(\breve{A} \vee_R \breve{C}\right)$

(iv) $\breve{A} \vee_R \left(\breve{B} \wedge_R \breve{C}\right) = \left(\breve{A} \vee_R \breve{B}\right) \wedge_R \left(\breve{A} \vee_R \breve{C}\right)$

(v) $\breve{A} \wedge_P \left(\breve{B} \vee_P \breve{C}\right) = \left(\breve{A} \wedge_P \breve{B}\right) \vee_P \left(\breve{A} \wedge_P \breve{C}\right)$

(vi) $\breve{A} \wedge_P \left(\breve{B} \vee_R \breve{C}\right) = \left(\breve{A} \wedge_P \breve{B}\right) \vee_R \left(\breve{A} \wedge_P \breve{C}\right)$

(vii) $\breve{A} \wedge_R \left(\breve{B} \vee_P \breve{C}\right) = \left(\breve{A} \wedge_R \breve{B}\right) \vee_P \left(\breve{A} \wedge_R \breve{C}\right)$

(viii) $\breve{A} \wedge_R \left(\breve{B} \vee_R \breve{C}\right) = \left(\breve{A} \wedge_R \breve{B}\right) \vee_R \left(\breve{A} \wedge_R \breve{C}\right)$

Proof. Let $\breve{A} = \left\langle \tilde{A}_{ij}^{a}, \lambda_{ij}^{a} \right\rangle, \breve{B} = \left\langle \tilde{B}_{ij}^{b}, \mu_{ij}^{b} \right\rangle, \breve{C} = \left\langle \tilde{C}_{ij}^{c}, \gamma_{ij}^{c} \right\rangle \in ICSM_{m\times n}$.

Then (i) $\breve{A} \vee_P \left(\breve{B} \wedge_P \breve{C}\right) = \left\langle \tilde{A}_{ij}^{a}, \lambda_{ij}^{a} \right\rangle \vee_P \left(\left\langle \tilde{B}_{ij}^{b} \mu_{ij}^{b} \right\rangle \wedge_P \left\langle \tilde{C}_{ij}^{c}, \gamma_{ij}^{c} \right\rangle\right)$

$$\begin{aligned}
\breve{A} \vee_P \left(\breve{B} \wedge_P \breve{C}\right) &= \breve{A} \vee_P \left\langle \min\{\tilde{B}_{ij}^{b}, \tilde{C}_{ij}^{c}\}, \min\{\mu_{ij}^{b}, \gamma_{ij}^{c}\} \right\rangle \text{ for all i, j} \\
&= \left\langle \max\{\tilde{A}_{ij}^{a}, \min\{\tilde{B}_{ij}^{b}, \tilde{C}_{ij}^{c}\}\}, \max\{\lambda_{ij}^{a}, \min\{\mu_{ij}^{b}, \gamma_{ij}^{c}\}\} \right\rangle \text{ for all i, j} \\
&= \left\langle \min\{\max\{\tilde{A}_{ij}^{a}, \tilde{B}_{ij}^{b}\}, \max\{\tilde{A}_{ij}^{a}, \tilde{C}_{ij}^{c}\}\}, \min\{\max\{\lambda_{ij}^{a}, \mu_{ij}^{b}\}, \max\{\lambda_{ij}^{a}, \gamma_{ij}^{c}\}\} \right\rangle \\
&= \left(\breve{A} \vee_P \breve{B}\right) \wedge_P \left(\breve{A} \vee_P \breve{C}\right)
\end{aligned}$$

(ii) $\breve{A} \vee_P \left(\breve{B} \wedge_R \breve{C}\right) = \left\langle \tilde{A}_{ij}^{a}, \lambda_{ij}^{a} \right\rangle \vee_P \left(\left\langle \tilde{B}_{ij}^{b} \mu_{ij}^{b} \right\rangle \wedge_R \left\langle \tilde{C}_{ij}^{c}, \gamma_{ij}^{c} \right\rangle\right)$

$$\begin{aligned}
\breve{A} \vee_P \left(\breve{B} \wedge_R \breve{C}\right) &= \breve{A} \vee_P \left\langle \min\{\tilde{B}_{ij}^{b}, \tilde{C}_{ij}^{c}\}, \max\{\mu_{ij}^{b}, \gamma_{ij}^{c}\} \right\rangle \text{ for all i, j} \\
&= \left\langle \max\{\tilde{A}_{ij}^{a}, \min\{\tilde{B}_{ij}^{b}, \tilde{C}_{ij}^{c}\}\}, \max\{\lambda_{ij}^{a}, \max\{\mu_{ij}^{b}, \gamma_{ij}^{c}\}\} \right\rangle \text{ for all i, j} \\
&= \left\langle \min\{\max\{\tilde{A}_{ij}^{a}, \tilde{B}_{ij}^{b}\}, \max\{\tilde{A}_{ij}^{a}, \tilde{C}_{ij}^{c}\}\}, \max\{\max\{\lambda_{ij}^{a}, \mu_{ij}^{b}\}, \max\{\lambda_{ij}^{a}, \gamma_{ij}^{c}\}\} \right\rangle \\
&= \left(\breve{A} \vee_P \breve{B}\right) \wedge_R \left(\breve{A} \vee_P \breve{C}\right)
\end{aligned}$$

(iii) $\breve{A} \vee_R \left(\breve{B} \wedge_P \breve{C}\right) = \left\langle \tilde{A}_{ij}^{a}, \lambda_{ij}^{a} \right\rangle \vee_R \left(\left\langle \tilde{B}_{ij}^{b} \mu_{ij}^{b} \right\rangle \wedge_P \left\langle \tilde{C}_{ij}^{c}, \gamma_{ij}^{c} \right\rangle\right)$

$$\begin{aligned}
\breve{A} \vee_R \left(\breve{B} \wedge_P \breve{C}\right) &= \breve{A} \vee_R \left\langle \min\{\tilde{B}_{ij}^{b}, \tilde{C}_{ij}^{c}\}, \min\{\mu_{ij}^{b}, \gamma_{ij}^{c}\} \right\rangle \text{ for all i, j} \\
&= \left\langle \max\{\tilde{A}_{ij}^{a}, \min\{\tilde{B}_{ij}^{b}, \tilde{C}_{ij}^{c}\}\}, \min\{\lambda_{ij}^{a}, \min\{\mu_{ij}^{b}, \gamma_{ij}^{c}\}\} \right\rangle \text{ for all i, j} \\
&= \left\langle \min\{\max\{\tilde{A}_{ij}^{a}, \tilde{B}_{ij}^{b}\}, \max\{\tilde{A}_{ij}^{a}, \tilde{C}_{ij}^{c}\}\}, \min\{\min\{\lambda_{ij}^{a}, \mu_{ij}^{b}\}, \min\{\lambda_{ij}^{a}, \gamma_{ij}^{c}\}\} \right\rangle \\
&= \left(\breve{A} \vee_R \breve{B}\right) \wedge_P \left(\breve{A} \vee_R \breve{C}\right)
\end{aligned}$$

(iv) $\breve{A} \vee_R \left(\breve{B} \wedge_R \breve{C}\right) = \langle \tilde{A}^a_{ij}, \lambda^a_{ij} \rangle \vee_R \left(\langle \tilde{B}^b_{ij} \mu^b_{ij} \rangle \wedge_R \langle \tilde{C}^c_{ij}, \gamma^c_{ij} \rangle \right)$

$$\begin{aligned}\breve{A} \vee_R \left(\breve{B} \wedge_R \breve{C}\right) &= \breve{A} \vee_R \langle \min\{\tilde{B}^b_{ij}, \tilde{C}^c_{ij}\}, \max\{\mu^b_{ij}, \gamma^c_{ij}\} \rangle \text{ for all i, j} \\ &= \langle \max\{\tilde{A}^a_{ij}, \min\{\tilde{B}^b_{ij}, \tilde{C}^c_{ij}\}\}, \min\{\lambda^a_{ij}, \max\{\mu^b_{ij}, \gamma^c_{ij}\}\} \rangle \text{ for all i, j} \\ &= \langle \min\{\max\{\tilde{A}^a_{ij}, \tilde{B}^b_{ij}\}, \max\{\tilde{A}^a_{ij}, \tilde{C}^c_{ij}\}\}, \max\{\min\{\lambda^a_{ij}, \mu^b_{ij}\}, \min\{\lambda^a_{ij}, \gamma^c_{ij}\}\} \rangle \\ &= \left(\breve{A} \vee_R \breve{B}\right) \wedge_R \left(\breve{A} \vee_R \breve{C}\right)\end{aligned}$$

(v) $\breve{A} \wedge_P \left(\breve{B} \vee_P \breve{C}\right) = \langle \tilde{A}^a_{ij}, \lambda^a_{ij} \rangle \wedge_P \left(\langle \tilde{B}^b_{ij} \mu^b_{ij} \rangle \vee_P \langle \tilde{C}^c_{ij}, \gamma^c_{ij} \rangle \right)$

$$\begin{aligned}\breve{A} \wedge_P \left(\breve{B} \vee_P \breve{C}\right) &= \breve{A} \wedge_P \langle \max\{\tilde{B}^b_{ij}, \tilde{C}^c_{ij}\}, \max\{\mu^b_{ij}, \gamma^c_{ij}\} \rangle \text{ for all i, j} \\ &= \langle \min\{\tilde{A}^a_{ij}, \max\{\tilde{B}^b_{ij}, \tilde{C}^c_{ij}\}\}, \min\{\lambda^a_{ij}, \max\{\mu^b_{ij}, \gamma^c_{ij}\}\} \rangle \text{ for all i, j} \\ &= \langle \max\{\min\{\tilde{A}^a_{ij}, \tilde{B}^b_{ij}\}, \min\{\tilde{A}^a_{ij}, \tilde{C}^c_{ij}\}\}, \max\{\min\{\lambda^a_{ij}, \mu^b_{ij}\}, \min\{\lambda^a_{ij}, \gamma^c_{ij}\}\} \rangle \\ &= \left(\breve{A} \wedge_P \breve{B}\right) \vee_P \left(\breve{A} \wedge_P \breve{C}\right)\end{aligned}$$

(vi) $\breve{A} \wedge_P \left(\breve{B} \vee_R \breve{C}\right) = \langle \tilde{A}^a_{ij}, \lambda^a_{ij} \rangle \wedge_P \left(\langle \tilde{B}^b_{ij} \mu^b_{ij} \rangle \vee_R \langle \tilde{C}^c_{ij}, \gamma^c_{ij} \rangle \right)$

$$\begin{aligned}\breve{A} \wedge_P \left(\breve{B} \vee_R \breve{C}\right) &= \breve{A} \wedge_P \langle \max\{\tilde{B}^b_{ij}, \tilde{C}^c_{ij}\}, \min\{\mu^b_{ij}, \gamma^c_{ij}\} \rangle \text{ for all i, j} \\ &= \langle \min\{\tilde{A}^a_{ij}, \max\{\tilde{B}^b_{ij}, \tilde{C}^c_{ij}\}\}, \min\{\lambda^a_{ij}, \min\{\mu^b_{ij}, \gamma^c_{ij}\}\} \rangle \text{ for all i, j} \\ &= \langle \max\{\min\{\tilde{A}^a_{ij}, \tilde{B}^b_{ij}\}, \min\{\tilde{A}^a_{ij}, \tilde{C}^c_{ij}\}\}, \min\{\min\{\lambda^a_{ij}, \mu^b_{ij}\}, \min\{\lambda^a_{ij}, \gamma^c_{ij}\}\} \rangle \\ &= \left(\breve{A} \wedge_P \breve{B}\right) \vee_R \left(\breve{A} \wedge_P \breve{C}\right)\end{aligned}$$

(vii) $\breve{A} \wedge_R \left(\breve{B} \vee_P \breve{C}\right) = \langle \tilde{A}^a_{ij}, \lambda^a_{ij} \rangle \wedge_R \left(\langle \tilde{B}^b_{ij} \mu^b_{ij} \rangle \vee_P \langle \tilde{C}^c_{ij}, \gamma^c_{ij} \rangle \right)$

$$\begin{aligned}\breve{A} \wedge_R \left(\breve{B} \vee_P \breve{C}\right) &= \breve{A} \wedge_R \langle \max\{\tilde{B}^b_{ij}, \tilde{C}^c_{ij}\}, \max\{\mu^b_{ij}, \gamma^c_{ij}\} \rangle \text{ for all i, j} \\ &= \langle \min\{\tilde{A}^a_{ij}, \max\{\tilde{B}^b_{ij}, \tilde{C}^c_{ij}\}\}, \max\{\lambda^a_{ij}, \max\{\mu^b_{ij}, \gamma^c_{ij}\}\} \rangle \text{ for all i, j} \\ &= \langle \max\{\min\{\tilde{A}^a_{ij}, \tilde{B}^b_{ij}\}, \min\{\tilde{A}^a_{ij}, \tilde{C}^c_{ij}\}\}, \max\{\max\{\lambda^a_{ij}, \mu^b_{ij}\}, \max\{\lambda^a_{ij}, \gamma^c_{ij}\}\} \rangle \\ &= \left(\breve{A} \wedge_R \breve{B}\right) \vee_P \left(\breve{A} \wedge_R \breve{C}\right)\end{aligned}$$

(viii) $\breve{A} \wedge_R \left(\breve{B} \vee_R \breve{C}\right) = \langle \tilde{A}^a_{ij}, \lambda^a_{ij} \rangle \wedge_R \left(\langle \tilde{B}^b_{ij} \mu^b_{ij} \rangle \vee_R \langle \tilde{C}^c_{ij}, \gamma^c_{ij} \rangle \right)$

$$\begin{aligned}\breve{A} \wedge_R \left(\breve{B} \vee_R \breve{C}\right) &= \breve{A} \wedge_R \langle \max\{\tilde{B}^b_{ij}, \tilde{C}^c_{ij}\}, \min\{\mu^b_{ij}, \gamma^c_{ij}\} \rangle \text{ for all i, j} \\ &= \langle \min\{\tilde{A}^a_{ij}, \max\{\tilde{B}^b_{ij}, \tilde{C}^c_{ij}\}\}, \max\{\lambda^a_{ij}, \min\{\mu^b_{ij}, \gamma^c_{ij}\}\} \rangle \text{ for all i, j} \\ &= \langle \max\{\min\{\tilde{A}^a_{ij}, \tilde{B}^b_{ij}\}, \min\{\tilde{A}^a_{ij}, \tilde{C}^c_{ij}\}\}, \min\{\max\{\lambda^a_{ij}, \mu^b_{ij}\}, \max\{\lambda^a_{ij}, \gamma^c_{ij}\}\} \rangle \\ &= \left(\breve{A} \wedge_R \breve{B}\right) \vee_R \left(\breve{A} \wedge_R \breve{C}\right)\end{aligned}$$

Proposition 3.13 *Let* $\breve{A} \in ICSM_{m\times n}(resp.ECSM_{m\times n})$. *Then*

(i) $\left(\breve{A} \vee_P \breve{A}\right)^T = \breve{A}$.

(ii) $\left(\breve{A} \vee_R \breve{A}\right)^T = \breve{A}$.

(iii) $\left(\breve{A} \wedge_P \breve{A}\right)^T = \breve{A}$.

(iv) $\left(\breve{A} \wedge_R \breve{A}\right)^T = \breve{A}$.

(v) $\left(\breve{A}^T\right)^T = \breve{A}$.

Proof. Let $\breve{A} = \langle \tilde{A}^a_{ij}, \lambda^a_{ij} \rangle \in ICSM_{m\times n}$, and $\breve{A}^T = \langle \tilde{A}^a_{ji}, \lambda^a_{ji} \rangle$.

(i) $\left(\breve{A} \vee_P \breve{A}\right)^T = \left(\left\langle \tilde{A}^a_{ij}, \lambda^a_{ij} \right\rangle \vee_P \left\langle \tilde{A}^a_{ij}, \lambda^a_{ij} \right\rangle\right)^T$ for all i, j.

$$\begin{aligned}\left(\breve{A} \vee_P \breve{A}\right)^T &= \left(\left\langle \max\{\tilde{A}^a_{ij}, \tilde{A}^a_{ij}\}, \max\{\lambda^a_{ij}, \lambda^a_{ij}\}\right\rangle\right)^T \\ &= \left(\left\langle \tilde{A}^a_{ij}, \lambda^a_{ij} \right\rangle\right)^T \\ &= \left\langle \tilde{A}^a_{ji}, \lambda^a_{ji} \right\rangle \\ &= \left(\breve{A}\right)^T.\end{aligned}$$

(ii) $\left(\breve{A} \vee_R \breve{A}\right)^T = \left(\left\langle \tilde{A}^a_{ij}, \lambda^a_{ij} \right\rangle \vee_R \left\langle \tilde{A}^a_{ij}, \lambda^a_{ij} \right\rangle\right)^T$ for all i, j.

$$\begin{aligned}\left(\breve{A} \vee_R \breve{A}\right)^T &= \left(\left\langle \max\{\tilde{A}^a_{ij}, \tilde{A}^a_{ij}\}, \min\{\lambda^a_{ij}, \lambda^a_{ij}\}\right\rangle\right)^T \\ &= \left(\left\langle \tilde{A}^a_{ij}, \lambda^a_{ij} \right\rangle\right)^T \\ &= \left\langle \tilde{A}^a_{ji}, \lambda^a_{ji} \right\rangle \\ &= \left(\breve{A}\right)^T.\end{aligned}$$

(iii) $\left(\breve{A} \wedge_P \breve{A}\right)^T = \left(\left\langle \tilde{A}^a_{ij}, \lambda^a_{ij} \right\rangle \vee_P \left\langle \tilde{A}^a_{ij}, \lambda^a_{ij} \right\rangle\right)^T$ for all i, j.

$$\begin{aligned}\left(\breve{A} \wedge_P \breve{A}\right)^T &= \left(\left\langle \min\{\tilde{A}^a_{ij}, \tilde{A}^a_{ij}\}, \min\{\lambda^a_{ij}, \lambda^a_{ij}\}\right\rangle\right)^T \\ &= \left(\left\langle \tilde{A}^a_{ij}, \lambda^a_{ij} \right\rangle\right)^T \\ &= \left\langle \tilde{A}^a_{ji}, \lambda^a_{ji} \right\rangle \\ &= \left(\breve{A}\right)^T.\end{aligned}$$

(iv) $\left(\breve{A} \wedge_R \breve{A}\right)^T = \left(\left\langle \tilde{A}^a_{ij}, \lambda^a_{ij} \right\rangle \wedge_R \left\langle \tilde{A}^a_{ij}, \lambda^a_{ij} \right\rangle\right)^T$ for all i, j.

$$\begin{aligned}\left(\breve{A} \wedge_R \breve{A}\right)^T &= \left(\left\langle \min\{\tilde{A}^a_{ij}, \tilde{A}^a_{ij}\}, \max\{\lambda^a_{ij}, \lambda^a_{ij}\}\right\rangle\right)^T \\ &= \left(\left\langle \tilde{A}^a_{ij}, \lambda^a_{ij} \right\rangle\right)^T \\ &= \left\langle \tilde{A}^a_{ji}, \lambda^a_{ji} \right\rangle \\ &= \left(\breve{A}\right)^T.\end{aligned}$$

(v) $\left(\breve{A}^T\right)^T = \left(\left\langle \tilde{A}^a_{ij}, \lambda^a_{ij} \right\rangle^T\right)^T$ for all i, j.

$$\begin{aligned}\left(\breve{A} \wedge_R \breve{A}\right)^T &= \left(\left\langle \tilde{A}^a_{ji}, \lambda^a_{ji} \right\rangle\right)^T \\ &= \left\langle \tilde{A}^a_{ij}, \lambda^a_{ij} \right\rangle \\ &= \breve{A}.\end{aligned}$$

Proposition 3.14 *Let* $\breve{A}, \breve{B} \in ICSM_{m\times n}(resp.ECSM_{m\times n})$. *Then*

(i) $\left(\breve{A} \vee_P \breve{B}\right)^T = \left(\breve{A}\right)^T \vee_P \left(\breve{B}\right)^T$.

(ii) $\left(\breve{A} \vee_R \breve{B}\right)^T = \left(\breve{A}\right) \vee_R \left(\breve{B}\right)^T$.

(iii) $\left(\breve{A} \wedge_P \breve{B}\right)^T = \left(\breve{A}\right) \wedge_P \left(\breve{B}\right)^T$.

(iv) $\left(\breve{A} \wedge_R \breve{B}\right)^T = \left(\breve{A}\right) \wedge_R \left(\breve{B}\right)^T$.

Proof. Let $\breve{A}, \breve{B} \in ICSM_{m\times n}(resp.ECSM_{m\times n})$.

(i) $\left(\breve{A} \vee_P \breve{B}\right)^T = \left(\left\langle \tilde{A}^a_{ij}, \lambda^a_{ij} \right\rangle \vee_P \left\langle \tilde{B}^b_{ij}, \mu^b_{ij} \right\rangle\right)^T$ for all i, j.

$$\begin{aligned} \left(\breve{A} \vee_P \breve{B}\right)^T &= \left(\left\langle \max\{\tilde{A}^a_{ij}, \tilde{B}^b_{ij}\}, \max\{\lambda^a_{ij}, \mu^b_{ij}\} \right\rangle\right)^T \text{ for all i, j} \\ &= \left\langle \max\{\tilde{A}^a_{ji}, \tilde{B}^b_{ji}\}, \max\{\lambda^a_{ji}, \mu^b_{ji}\} \right\rangle \text{ for all i, j} \\ &= \left(\breve{A}\right)^T \vee_P \left(\breve{B}\right)^T. \end{aligned}$$

(ii) $\left(\breve{A} \vee_R \breve{B}\right)^T = \left(\left\langle \tilde{A}^a_{ij}, \lambda^a_{ij} \right\rangle \vee_R \left\langle \tilde{B}^b_{ij}, \mu^b_{ij} \right\rangle\right)^T$ for all i, j.

$$\begin{aligned} \left(\breve{A} \vee_R \breve{B}\right)^T &= \left(\left\langle \max\{\tilde{A}^a_{ij}, \tilde{B}^b_{ij}\}, \min\{\lambda^a_{ij}, \mu^b_{ij}\} \right\rangle\right)^T \text{ for all i, j} \\ &= \left\langle \max\{\tilde{A}^a_{ji}, \tilde{B}^b_{ji}\}, \min\{\lambda^a_{ji}, \mu^b_{ji}\} \right\rangle \text{ for all i, j} \\ &= \left(\breve{A}\right)^T \vee_R \left(\breve{B}\right)^T. \end{aligned}$$

(iii) $\left(\breve{A} \wedge_P \breve{B}\right)^T = \left(\left\langle \tilde{A}^a_{ij}, \lambda^a_{ij} \right\rangle \wedge_P \left\langle \tilde{B}^b_{ij}, \mu^b_{ij} \right\rangle\right)^T$ for all i, j.

$$\begin{aligned} \left(\breve{A} \wedge_P \breve{B}\right)^T &= \left(\left\langle \min\{\tilde{A}^a_{ij}, \tilde{B}^b_{ij}\}, \min\{\lambda^a_{ij}, \mu^b_{ij}\} \right\rangle\right)^T \text{ for all i, j} \\ &= \left\langle \min\{\tilde{A}^a_{ji}, \tilde{B}^b_{ji}\}, \min\{\lambda^a_{ji}, \mu^b_{ji}\} \right\rangle \text{ for all i, j} \\ &= \left(\breve{A}\right)^T \wedge_P \left(\breve{B}\right)^T. \end{aligned}$$

(iv) $\left(\breve{A} \wedge_R \breve{B}\right)^T = \left(\left\langle \tilde{A}^a_{ij}, \lambda^a_{ij} \right\rangle \wedge_R \left\langle \tilde{B}^b_{ij}, \mu^b_{ij} \right\rangle\right)^T$ for all i, j.

$$\begin{aligned} \left(\breve{A} \wedge_R \breve{B}\right)^T &= \left(\left\langle \min\{\tilde{A}^a_{ij}, \tilde{B}^b_{ij}\}, \max\{\lambda^a_{ij}, \mu^b_{ij}\} \right\rangle\right)^T \text{ for all i, j} \\ &= \left\langle \min\{\tilde{A}^a_{ji}, \tilde{B}^b_{ji}\}, \max\{\lambda^a_{ji}, \mu^b_{ji}\} \right\rangle \text{ for all i, j} \\ &= \left(\breve{A}\right)^T \wedge_R \left(\breve{B}\right)^T. \end{aligned}$$

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