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MAJORIZATION OF MATRICES

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Abstract:

It is shown that under certain conditions the column majorization of matrices is reversed for the column majorization of their corresponding Moore-Penrose inverses and preserved for the column majorization of their powers. The condition for column majorization of block matrices is determined.

Index Terms: Majorization & Moore-Penrose inverse

1. Introduction:

Let $C^{m\times n}$ denote the space of $m\times n$ complex matrices. If $A\in C^{m\times n}$ then the Moore-Penrose inverse A^+ of A is the unique solution to the equations:

$$AXA = A,\ XAX = X,\ (AX)^* = AX \text{ and } (XA)^* = XA\ [2, p.7].$$

A square matrix A is called EP if $R(A) = R(A^*)$ or equivalently $AA^+ = A^+A$, where $R(A)$ denotes the range space of A.A matrix A is called EP_r if A is EP and is of rank r.

Let $R^{m\times n}$ denotes the space of $m\times n$ real matrices. For any column vector $x\in R^n$, let $x_{[1]}, x_{[2]},\ldots x_{[n]}$ denote the coordinates of x arranged in decreasing order of magnitude: $x_{[1]}\geq x_{[2]}\geq \ldots\geq x_{[n]}$. We shall write $x\downarrow=(x_{[1]}, x_{[2]},..x_{[n]})^t$ where t denotes the transpose. If $x, y\in R^n$, we say that y is majorized by x, denote $y\prec x$ if

$$\sum_{i=1}^{k} y_i = \sum_{i=1}^{k} x_i,\quad for\ 1\leq k\leq n-1, \text{ and } \sum_{i=1}^{n} y_i = \sum_{i=1}^{n} x_i$$

Also y is said to be majorized by x if and only if there exists a doubly stochastic matrix M such that $y = Mx$ [4, p.7-12]. Throughout this paper we consider only real matrices.

2. Majorization of Matrices:

The majorization of vectors is extended to matrices as follows:

Definition 1:

Let A and B be $m\times n$ real matrices. Then A is said to be column majorized by B, denoted by $A\prec^c B$ if and only if $A = MB$ where M is a doubly stochastic matrix of order m.

We note that the column majorization of matrices is equivalent to the majorization of the transpose of the corresponding matrices [4, p.430].

i.e. $A\prec^c B \Leftrightarrow A = MB$, where M is doubly stochastic

$\Leftrightarrow A = MB$ for all i, where A_i is the i^{th} column of A

$\Leftrightarrow A_i^t = B_i^t$ $for\ all\ i$

$\Leftrightarrow A^t \prec B^t$.

Lemma 1:

Let A, B be EP_r matrices and $A\prec^c B$. Then $R(A) = R(B)$ and $(AB)^+ = B^+A^+$.

Proof:

$A\prec^c B \Rightarrow A = MB \Rightarrow N(B)\subseteq N(A)$ $where$ $N(A)$, denotes the null space of A.

Since A and B are EP_r, $R(A) = R(B)$, then by theorem 3 of [1], AB is EP_r and by theorem 4 of [1], we get $(AB)^+ = B^+A^+$.

Lemma 2:

Let A be an EP matrix. Then

$(i)\ A \prec^c A^+A \Leftrightarrow A^+A \prec^c A^+$

$(ii)\ A^+A \prec^c A \Leftrightarrow A^+ \prec^c A^+A$

Proof:

$$. (i)\ A \prec^c A^+A \Leftrightarrow A = MA^+A$$
$$\Leftrightarrow AA^+ = MA^+AA^+ \ (on\ post\ multiplication\ by\ A^+\ or\ A)$$
$$\Leftrightarrow AA^+ = MA^+ \ (\text{since}\ A\ is\ EP\ and\ A^+AA^+ = A^+)$$
$$\Leftrightarrow A^+A \prec^c A^+$$

Hence the result (i). Similarly (ii) can be proved.

Remark 1:

In particular, if A is nonsingular, then $A \prec^c I \Leftrightarrow I \prec^c A^{-1}$. Hence $A \prec^c I \Rightarrow A \prec^c A^{-1}$. If the column sums of A is one, then $I \prec^c A^{-1}$, however A need not be a doubly stochastic matrix.

Remark 2:

We note that the condition on A cannot be relaxed in the above lemma 2. For example,

$$A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \qquad A^+ = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$
$$AA^+ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad A^+A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Here A is not EP, however $A \prec^c A^+A$ and $A^+A \prec^c A^+$.

Remark 3:

For any EP matrix A, if $A \prec^c I$, then $A = AA^+A \prec^c A^+A$. Hence by lemma 2 $A \prec^c A^+A \prec^c A^+$ In particular if A is EP and doubly stochastic, then $A \prec^c I$ holds automatically and hence $A \prec^c A^+A \prec^c A^+$

Theorem 1:

If A is EP_r and B is symmetric idempotent with rank r, then

$$(i)\ A \prec^c B \Leftrightarrow B^+ \prec^c A^+$$
$$(ii)\ B \prec^c A \Leftrightarrow A^+ \prec^c B^+$$
$$(iii)\ A \prec^c B \Rightarrow A \prec^c A^+$$
$$(iv)\ B \prec^c A \Rightarrow A^+ \prec^c A$$

Proof:

Since A is EP_r and B is symmetric idempotent with rank r, both A and B are EP_r. By lemma 1, $R(A) = R(B) \Rightarrow A^+A = B^+B = B$ (since A^+A is the projection onto $R(A)$ along $N(A)$. Then (i) and (ii) follow from lemma 2. (iii) and (iv) follow from (i) and (ii) respectively and $B = B^+$.

Corollary 1:

Let A be EP_r, B be symmetric idempotent with rank r and $A \prec^c B$. Then $A^n \prec^c B \prec^c (A^n)^+$ for any positive integer n.

Proof:

Since $A \prec^c B$ and by theorem 1 (i), $B = B^+ \prec^c A^+$. Hence, $A \prec^c A^+$ follows from theorem 1 (iii). SinceB is symmetric idempotent and by theoremB2 of [4p.433],

$A^2 \prec^c A^\dagger A = B^\dagger B = B$. Thus, $A^n \prec^c B$ is true for n=2. Now, $A^3 \prec^c BA = BB^\dagger A = AA^\dagger A = A \prec^c B$. Hence $A \prec^c B$. Thus , it is true for n=3. By continuing in this manner we can show that $A \prec^c B \Rightarrow A^n \prec^c B$ for any positive integer n. $B \prec^c (A^n)^\dagger$ follows from $A^n \prec^c B$ and theorem 1(i). Hence the corollary.

Theorem 2:

Let A and B be EP_r matrices such that $AB = BA$. Then $A \prec^c B \Leftrightarrow B^+ \prec^c A^+$.

Proof:

Since A, B are EP_r and $A \prec^c B$, by lemma 1, $R(A) = R(B) = R(B^+)$. Since A and B^+ are EP_r and $R(A) = R(B^+)$, by theorem 3 and 4 of [1], AB^+ is EP_r and $(AB^+)^+ = BA^+$

Thus, $AB^+ = B^+A$ (by theorem 2 of [5]) (1)

$A \prec^c B \Rightarrow AB^+ \prec^c BB^+$ (by theorem $B2$ of [4,p.433])

$\Rightarrow B^+A \prec^c BB^+$ (by (1))

$\Rightarrow BB^+ \prec^c (B^+A)^+$ (by theorem 1(i) applied to BB^+)

$\Rightarrow B^+B \prec^c A^+B$

$\Rightarrow B^+BB^+ \prec^c A^+BB^+$

$\Rightarrow B^+ \prec^c A^+$

Conversely, since A^+ and B^+ are EP_r. And by theorem 2 of [5], $BA^+ = A^+B$ is of rank r. In the above part, replacing A by B^+ and B by A^+ and using $(A^+)^+ = A, (B^+)^+ = B$,

We get $B^+ \prec^c A^+ \Rightarrow A \prec^c B$. Hence the theorem.

REMARK 4:

The condition on A and B that they have the same rank, but $AB \neq BA$ cannot hold in theorem 2. For example, consider,

$$A = 1/3 \begin{pmatrix} 12 & 3 & 3 \\ 11 & 12 & 10 \\ 7 & 12 & 11 \end{pmatrix} : \quad B = \begin{pmatrix} 5 & 0 & 0 \\ 3 & 6 & 5 \\ 2 & 3 & 3 \end{pmatrix}$$

$$A^+ = 1/45 \begin{pmatrix} 12 & 3 & -6 \\ -51 & 111 & -87 \\ 48 & -123 & 111 \end{pmatrix}; B^+ = 1/15 \begin{pmatrix} 3 & 0 & 0 \\ 1 & 15 & -25 \\ -3 & -15 & 30 \end{pmatrix}$$

Here $AB \neq BA$, however $A \prec^c B$, A and B are EP_r with rank 3 and $B^+ \prec^c A^+$.

Corollary 2:

Let A and B be EP_r matrices and $A \prec^c B$ such that $AB = BA$. Then $A^n \prec^c B^n$ for any positive integer n.

Proof:

By theorem $B2$ of [4, p.433] and $A \prec^c B$ we have $AB^+ \prec^c BB^+$. Since BB^+ is symmetric idempotent by theorem 1 (iii), we get

$$AB^+ \prec^c (AB^+)^+ = BA^+. \text{ By lemma 1, } AB^+A \prec^c BA^+A = B.$$

Using (1), we see that $AAB^+ \prec^c B$. By lemma 1, $A^+A = B^+B$.

Hence $A^2B^+B \prec^c B^2 \Rightarrow A^2 \prec^c B^2$. Thus the corollary is true for n=2.

Now, $A^2 \prec^c B^2 \Rightarrow A^3 \prec^c B^2A = AB^2 \prec^c BB^2 = B^3$

(Since $A \prec^c B \Rightarrow AB^2 \prec^c BB^2$). Thus it is true for $n = 3$. By continuing in this manner, we can show that $A \prec^c B \Rightarrow A^n \prec^c B^n$ for any positive integer n. Hence the corollary.

Theorem 3:

Let $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $rank\ M = rank\ A$ and $L = \begin{pmatrix} E & F \\ G & H \end{pmatrix}$ with

$rank\ L = rank\ E$ be $n \times n$ real matrices such that $A^{+}B = E^{+}F$ (2)

Then $A \prec^{c} E$ and $C \prec^{c} G$ .i.e., $A = R_1 E$ and $C = R_2 G \Leftrightarrow M \prec^{c} L$ with doubly stochastic matrix R of the form $\begin{pmatrix} O & R_1 \\ O & R_2 \end{pmatrix}$

Proof:

Since $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$ with $rank\ M = rank\ A,$ where A is $k \times k.$ and D is $(n-k)\times(n-k)$ matrices, by corollary in [3] it follows that $N(A) \subseteq N(C),\ \ N(A^{*}) \subseteq N(B^{*})$ and $D = CA^{+}B$ or equivalently, $C = CA^{+}A,\ B = AA^{+}B$ and $D = CA^{+}B$.

For $L,$ we have $G = GE^{+}E,\ F = EE^{+}F$ and $H = GE^{+}F.$ (3)

$$\text{Now, } A \prec^{c} E \text{ and } C \prec^{c} G \Leftrightarrow \begin{pmatrix} A & O \\ C & O \end{pmatrix} \prec^{c} \begin{pmatrix} E & O \\ G & O \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} A & O \\ C & O \end{pmatrix} \begin{pmatrix} I & A^{+}B \\ O & I \end{pmatrix} \prec^{c} \begin{pmatrix} E & O \\ G & O \end{pmatrix} \begin{pmatrix} I & A^{+}B \\ O & I \end{pmatrix}$$

$$\Leftrightarrow \begin{pmatrix} A & B \\ C & D \end{pmatrix} \prec^{c} \begin{pmatrix} E & F \\ G & H \end{pmatrix}$$

$$\Leftrightarrow M \prec^{c} L$$

Hence the theorem.

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