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NUMERICAL SOLUTIONS FOR SOLVING FIVPS USING THIRD ORDER RUNGE-KUTTA METHOD BASED ON VARIETY OF MEANS

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Abstract:

This paper presents Runge-Kutta (RK) method based on Arithmetic Mean (AM), Geometric Mean (GM), Harmonic Mean (HaM), Heronian Mean (HeM), Root Mean Square (RM), Centroidal Mean (CeM) and Contraharmonic Mean (CoM) to solve the intuitionistic Fuzzy differential equations (IFDEs). The efficiency of these methods has been illustrated through numerical examples of intuitionistic fuzzy differential equations.

Key Words: Intuitionistic Fuzzy differential equations, IVP, RK, AM, GM, HaM, HeM, RM, CeM and CoM.

1. Introduction:

The concept of fuzzy set theory has been extended into intuitionistic fuzzy set (IFS) theory by Atanassov [2–4]. The fuzzy sets theory was considered to be one of Intuitionistic fuzzy set (IFS) theory. Now-a-days, IFSs are being studied widely used in different fields of biology, engineering, physics as well as among other field of science. The studies on improvement of IFS theory, together with intuitionistic fuzzy geometry, intuitionistic fuzzy logic, intuitionistic fuzzy topology, an intuitionistic fuzzy approach to artificial intelligence, and intuitionistic fuzzy generalized nets have been given in [14].

Many authors have contributed the numerical solution of fuzzy differential equations with intuitionistic fuzzy initial value problems as follows: Numerical solution of FDE by Runge-Kutta method with intuitionistic treatment treated by Abbasbandy and Allahviranloo [1]. Murugesan et al. have discussed a comparison of extended Runge Kutta formulae based on variety of means to solve system ofIVPs[16]. Melliani et al. [5-8] have discussed differential and partial differential equations under intuitionistic fuzzy environment. Sneh Lata and Amit Kumar [13] have introduced time dependent intuitionistic fuzzy linear differential equation. The solution of intuitionistic fuzzy ordinary differential equation and introduced second order linear differential equations using the fuzzy boundary value by Sankar Prasad Mondal and Tapan Kumar Roy in [9, 12]. The convergence and stability of fuzzy initial value problems discussed by Kelava [4] and Ma et al. [8].

Sneh Lata and Amit Kumar [13] have introduced time dependent intuitionistic fuzzy linear differential equation and have proposed a method to solve it. Nirmala and Chenthur Pandian have studied Intuitionistic fuzzy differential equation with initial condition by using Euler method [11]. Ettoussi et al. [7] has been discussed by the existence and uniqueness of a solution of the intuitionistic fuzzy differential equation using the method of successive approximation.

This paper presents RK methods based on variety of means such as AM, CeM, CoM, HaM, HeM, RM and GM for solving intuitionistic fuzzy IVPs. The efficiency of these methods has been illustrated by numerical examples. For completeness, the RK methods using variety of means for ODEs has also been provided.

2. Preliminaries:

Definition 1:

Let a set X be fixed. An IFS A in X is an object having the form

$A=\{<x,\mu_A(x),\vartheta_A(x)>0:x\in X\}$ where the $\mu_A(x):\mathrm{X}\to[0,1]$ and

$\vartheta_A(x):\mathrm{X}\to[0,1]$ define the degree of membership and degree of non-membership respectively, the element $x\in X$ to the set A which is a subset of X, for every element of $x\in X$, $0<\mu_A(x),\ \vartheta_A(x)\le 1$

Definition 2:

An IFN A is defined as follows

- an intuitionistic fuzzy subset of real line.
- normal. i.e., there is any $x_0\in R$ such that $\mu_A(x)=1$ (so $\vartheta_{\tilde{A}^i}(x)=0$)
- a convex set for the membership function $\mu_A(x)$ i.e.,

$$\mu_A(\lambda x_1 + (1-\lambda)x_2) \ge \min(\mu_A(x_1), \mu_A(x_2)) \; \forall x_1, x_2 \in R, \lambda \in [0,1]$$

- a concave set for the non-membership function $\vartheta_{\tilde{A}^i}(x)$, i.e.,

$$\vartheta_A(\lambda x_1 + (1-\lambda)x_2) \le \min(\vartheta_A(x_1), \vartheta_A(x_2)) \; \forall x_1, x_2 \in R, \lambda \in [0,1].$$

Definition 3:

The α, β-cut of an IFN $A = \{x, \mu_A(x), \vartheta_A(x) \setminus x \in X\}$ is defined as follows:

$$A = \{x, \mu_A(x), \vartheta_A(x) \setminus x \in X, \mu_A(x) \ge \alpha \, and \, \vartheta_A(x) \le 1-\alpha\} \; \forall x \in [0,1]$$

The α-cut representation of IFN A generates the following pair of intervals and is denoted by $[\mathrm{A}]_\alpha = \{[\mathrm{A}_L^+(\alpha), \mathrm{A}_U^+(\alpha)], [\mathrm{A}_L^-(\beta), \mathrm{A}_U^-(\beta)]\}$.

Definition 4:

An intuitionistic fuzzy set $A = \{x, \mu_A(x), \vartheta_A(x) \setminus x \in X\}$ such that $\mu_A(x)$ and $(1-\vartheta_A)(x) = 1 - \vartheta_A(x), \forall \mathrm{x} \in R$ are fuzzy numbers, is called an intuitionistic fuzzy number.

Therefore IFS $A = \{x, \mu_A(x), \vartheta_A(x) \setminus x \in X\}$ is a conjecture of two fuzzy numbers, A^+ with a membership function $\mu_{A^+}(x) = \mu_A(x)$ and A^- with a membership function $\mu_{A^-}(x) = 1 - \vartheta_A(x)$.

Definition 5:

A Triangular Intuitionistic Fuzzy Number (TIFN) A is an intuitionistic fuzzy set in R with the following membership function $\mu_A(x)$ and non-membership function $\vartheta_A(x)$. given as follows:

$$\mu_A(x) = \begin{cases} \dfrac{x-a_1}{a_2-a_1}, & a_1 \le x \le a_2 \\ \dfrac{a_3-x}{a_3-a_2}, & a_2 \le x \le a_3 \\ 0, & \text{otherwise} \end{cases} \qquad \vartheta_A(x) = \begin{cases} \dfrac{a_2-x}{a_2-a_1'}, & a_1' \le x \le a_2 \\ \dfrac{x-a_2}{a_3'-a_2}, & a_2 \le x \le a_3' \\ 1, & \text{otherwise} \end{cases}$$

Where $a_1' \le a_1 \le a_2 \le a_3 \le a_3'$ and TIFN is denoted by $A = (a_1, a_2, a_3; \, a_1', \, a_2, \, a_3')$

Arithmetic operators over TIFNs can be found in [8].

Definition 6:

For arbitrary $u, v \in E^n$, the quantity $\mathrm{D}(u, v) = \sup - 0 \le \alpha \le 1 d([\mathrm{u}]^\alpha, [\mathrm{v}]^\alpha)$ is the distance between u and v, where d is the Hausdroff metric in E^n.

Definition 7:

Let mapping $f : I \rightarrow W^n$ for some interval I be an intuitionistic fuzzy function. The $\alpha, \beta -$cut of f is given by $[f(\mathrm{t})]_{\alpha,\beta} = \{[\underline{f^+}(\mathrm{t};\alpha), \overline{f^+}(\mathrm{t};\alpha)], [\underline{f^-}(\mathrm{t};\beta), \overline{f^-}(\mathrm{t};\beta)]\}$,

Where

$$\underline{f^+}(\mathrm{t};\alpha) = \min\{f^+(\mathrm{t};\alpha) \mid \mathrm{t} \in \mathrm{I}, 0 \le \alpha \le 1\}, \quad \overline{f^+}(\mathrm{t};\alpha) = \max\{f^+(\mathrm{t};\alpha) \mid \mathrm{t} \in \mathrm{I}, 0 \le \alpha \le 1\},$$

$$\underline{f^-}(\mathrm{t};\alpha) = \min\{f^-(\mathrm{t};\beta) \mid \mathrm{t} \in \mathrm{I}, 0 \le \beta \le 1\}, \quad \overline{f^-}(\mathrm{t};\alpha) = \max\{f^-(\mathrm{t};\beta) \mid \mathrm{t} \in \mathrm{I}, 0 \le \beta \le 1\}.$$

3. Intuitionistic Fuzzy Cauchy Problem:

A first order intuitionistic fuzzy differential equation is a differential equation of the form

$$\begin{cases} y'(t) = f(t, y(t)), & t \in [\mathrm{a}, \mathrm{b}] \\ y(t_0) = y_0 \end{cases} \qquad (3.1)$$

Where y is an intuitionistic fuzzy function of the crisp variable t, $f(t, y(t))$ is an intuitionistic fuzzy function of the crisp variable t and the intuitionistic fuzzy variable y and y' is the intuitionistic fuzzy derivative. If an initial value $y(t_0) = y_0$ {intuitionistic fuzzy number}, we get an intuitionistic fuzzy Cauchy problem of first order $y'(t) = f(t, y(t)), \;\; y(t_0) = y_0$

As each intuitionistic fuzzy number is a conjecture two fuzzy numbers, equation (3.1) can be replaced by an equivalent system as follows:

$$y'(t)=\left\{\underline{y}'^{+}(t;\alpha),\overline{y}'^{+}(t;\alpha)],[\underline{y}'^{-}(t;\alpha),\overline{y}^{-}(t;\alpha)]\right\}$$, where

$$\underline{y}'^{+}(t;\alpha)=\underline{f}^{+}(t,y^{+})=\min\left\{f^{+}(t.u)\,|\,u\in[\underline{y}^{+},\overline{y}^{+}]\right\}$$
$$=F(t,\underline{y}^{+},\overline{y}^{+}),\quad \underline{y}^{+}(t_0)=\underline{y_0}^{+} \qquad (3.2)$$

$$\overline{y}'^{+}(t;\alpha)=\overline{f}^{+}(t,y^{+})=\max\left\{f^{+}(t.u)\,|\,u\in[\underline{y}^{+},\overline{y}^{+}]\right\}$$
$$=G(t,\underline{y}^{+},\overline{y}^{+}),\quad \overline{y}^{+}(t_0)=\overline{y_0}^{+} \qquad (3.3)$$

$$\underline{y}'^{-}(t;\alpha)=\underline{f}^{-}(t,y^{-})=\min\left\{f^{-}(t.u)\,|\,u\in[\underline{y}^{-},\overline{y}^{-}]\right\}$$
$$=H(t,\underline{y}^{-},\overline{y}^{-}),\quad \underline{y}^{-}(t_0)=\underline{y_0}^{-} \qquad (3.4)$$

$$\overline{y}'^{-}(t;\alpha)=\overline{f}^{-}(t,y^{-})=\max\left\{f^{-}(t.u)\,|\,u\in[\underline{y}^{-},\overline{y}^{-}]\right\}$$
$$=I(t,\underline{y}^{-},\overline{y}^{-}),\quad \overline{y}^{-}(t_0)=\overline{y_0}^{-} \qquad (3.5)$$

The system of equations given in (3.2) and (3.3) will have unique solution $[\underline{y}^{+},\overline{y}^{+}]\in B=\bar{c}[0,1]\times\bar{c}[0,1]$ and the system of equations given in (3.4) and(3.5) will have unique solution $[\underline{y}^{-},\overline{y}^{-}]\in B=\bar{c}[0,1]\times\bar{c}[0,1]$

Therefore, the system given from equations (3.2) to (3.5) possesses unique solution $y(t)=\left\{[\underline{y}^{+}(t),\overline{y}^{+}(t)],[\underline{y}^{-}(t),\overline{y}^{-}(t)]\right\}\in B\times B$ which is an intuitionistic fuzzy function.

(i. e) for each t, $y(t;\alpha)=\left\{[\underline{y}^{+}(t;\alpha),\overline{y}^{+}(t;\alpha)],[\underline{y}^{-}(t;\alpha),\overline{y}^{-}(t;\alpha)]\right\},\alpha\in[0,1]$ is an intuitionistic fuzzy number.

The parametric form of the system of equations (3.2) to (3.5) is given by

$$\underline{y}'^{+}(t;\alpha)=F(t,\underline{y}^{+}(t;\alpha),\overline{y}^{+}(t;\alpha)),\quad \underline{y}^{+}(t_0;\alpha)=\underline{y_0}^{+}(\alpha)$$
$$\overline{y}'^{+}(t;\alpha)=G(t,\underline{y}^{+}(t;\alpha),\overline{y}^{+}(t;\alpha)),\quad \overline{y}^{+}(t_0;\alpha)=\overline{y_0}^{+}(\alpha)$$
$$\underline{y}'^{-}(t;\alpha)=H(t,\underline{y}^{-}(t;\alpha),\overline{y}^{-}(t;\alpha)),\quad \underline{y}^{-}(t_0;\alpha)=\underline{y_0}^{-}(\alpha)$$
$$\overline{y}'^{-}(t;\alpha)=I(t,\underline{y}^{-}(t;\alpha),\overline{y}^{-}(t;\alpha)),\quad \overline{y}^{-}(t_0;\alpha)=\overline{y_0}^{-}(\alpha)$$

for $\alpha\in[0,1]$.

4. Third Order Runge-Kutta Methods for Solving Intuitionistic fuzzy Differential Equations:

A first order intuitionistic fuzzy differential equation is a differential equation of the form

$$y'=f(t,y),\qquad y(t_0)=y_0 \quad (4.1)$$

The construction of Runge-Kutta methods is of the form

$$y_{n+1}=y_n+\int_0^h f(t_n+\tau,y(t_n+\tau))\,d\tau \quad (4.2)$$

Then the integral in (6.2.2) is approximated by a quadrature formula can be written as

$$y_{n+1}=y_n+h\sum_{i=1}^{s}b_iK_i \quad (4.3a)$$

Where

$$K_i = f(t_n + c_i h,\ \ y_n + h\sum_{j=1}^{i-1} a_{ij} K_j),\quad i = 1, 2, \cdots, s \quad (4.3b)$$

Fourth order RK for intuitionistic fuzzy IVPs

$$\underline{y}^+(t_{n+1};\alpha) = \underline{y}^+(t_n;\alpha) + h\sum_{i=1}^{3} b_i K_i$$

$$\overline{y}^+(t_{n+1};\alpha) = \overline{y}^+(t_n;\alpha) + h\sum_{i=1}^{3} b_i L_i \quad (4.4a)$$

$$\underline{y}^-(t_{n+1};\beta) = \underline{y}^-(t_n;\beta) + h\sum_{i=1}^{3} b_i M_i \quad \overline{y}^-(t_{n+1};\beta) = \overline{y}^-(t_n;\beta) + h\sum_{i=1}^{3} b_i N_i$$

Where

$$K_i = F(t_n + c_i h,\ \ \underline{y}^+(t_n;\alpha) + h\sum_{j=1}^{i-1} a_j K_j),$$

$$L_i = G(t_n + c_i h,\ \ \overline{y}^+(t_n;\alpha) + h\sum_{j=1}^{i-1} a_j L_j),$$

$$M_i = H(t_n + c_i h,\ \ \underline{y}^+(t_n;\beta) + h\sum_{j=1}^{i-1} a_j M_j), \quad (4.4b)$$

$$N_i = I(t_n + c_i h,\ \ \overline{y}^+(t_n;\beta) + h\sum_{j=1}^{i-1} a_j N_j),\quad i = 1, 2, 3.$$

Where a_j's and c_i's are constants.

The values of AM can be replaced by variety of means such as GM, HaM, HeM, RM, CeM and CoM. For convenience, we take $a_1 = a_{11}$, $a_2 = a_{21}$and $a_3 = a_{32}$,. The missing elements in the matrix A = (a_{ij}), i,j=1,2,3 are defined to be zero. The formulae for HaM, HeM,RM, CeM and CoM in terms of AM and GM and also the values for $a_{ij}$is for these means can be referred in [ ]. The fourth order RK formulae based on various means for solving fuzzy IVPs are given in Table 4.1, Table 4.2, Table 4.3 and Table 4.4.

Table 4.1

S.No	Means	$\underline{y}^+_{n+1}$
1.	AM	$\underline{y}^+_n + \frac{h}{4}\left[K_1 + 2K_2 + K_3\right]$
2.	GM	$\underline{y}^+_n + \frac{h}{2}\left[\sqrt{\lvert K_1 K_2\rvert} + \sqrt{\lvert K_2 K_3\rvert}\right]$
3.	HaM	$\underline{y}^+_n + h\left[\frac{K_1 K_2}{K_1 + K_2} + \frac{K_2 K_3}{K_2 + K_3} + \frac{K_3 K_4}{K_3 + K_4}\right]$
4.	HeM	$\underline{y}^+_n + \frac{h}{6}\left[K_1 + 2K_2 + K_3 + \sqrt{\lvert K_1 K_2\rvert} + \sqrt{\lvert K_2 K_3\rvert}\right]$
5.	RM	$\underline{y}^+_n + \frac{h}{2}\left[\sqrt{\frac{K_1^2 + K_2^2}{2}} + \sqrt{\frac{K_2^2 + K_3^2}{2}}\right]$
6.	CeM	$\underline{y}^+_n + \frac{1}{3}h\left[\frac{K_1^2 + K_1K_2 + k_2^2}{K_1 + K_2} + \frac{K_2^2 + K_2K_3 + K_3^2}{K_2 + K_3}\right]$
7.	CoM	$\underline{y}^+_n + \frac{h}{2}\left[\frac{K_1^2 + K_2^2}{K_1 + K_2} + \frac{K_2^2 + K_3^2}{K_2 + K_3}\right]$

Table 4.2

S.No	**Means**	$\overline{y}^{+}_{n+1}$
1.	AM	$\overline{y}^{+}_{n} + \frac{h}{4}\left[L_1 + 2L_2 + L_3\right]$
2.	GM	$\overline{y}^{+}_{n} + \frac{h}{2}\left[\sqrt{\lvert L_1 L_2\rvert} + \sqrt{\lvert L_2 L_3\rvert}\right]$
3.	HaM	$\overline{y}^{+}_{n} + h\left[\frac{L_1 L_2}{L_1 + L_2} + \frac{L_2 L_3}{L_2 + L_3}\right]$
4.	HeM	$\overline{y}^{+}_{n} + \frac{h}{6}\left[L_1 + 2L_2 + L_3 + \sqrt{\lvert L_1 L_2\rvert} + \sqrt{\lvert L_2 L_3\rvert}\right]$
5.	RM	$\overline{y}^{+}_{n} + \frac{h}{2}\left[\sqrt{\frac{L_1^2 + L_2^2}{2}} + \sqrt{\frac{L_2^2 + L_3^2}{2}}\right]$
6.	CeM	$\overline{y}^{+}_{n} + \frac{1}{3}h\left[\frac{L_1^2 + L_1 L_2 + L_2^2}{L_1 + L_2} + \frac{L_2^2 + L_2 L_3 + k_3^2}{L_2 + L_3}\right]$
7.	CoM	$\underline{y}^{+}_{n} + \frac{h}{2}\left[\frac{L_1^2 + L_2^2}{L_1 + L_2} + \frac{L_2^2 + L_3^2}{L_2 + L_3}\right]$

Table 4.3

S.No	**Means**	$\underline{y}^{-}_{n+1}$
1.	AM	$\underline{y}^{-}_{n} + \frac{h}{4}\left[M_1 + 2M_2 + M_3\right]$
2.	GM	$\underline{y}^{-}_{n} + \frac{h}{2}\left[\sqrt{\lvert M_1 M_2\rvert} + \sqrt{\lvert M_2 M_3\rvert}\right]$
3.	HaM	$\underline{y}^{-}_{n} + h\left[\frac{M_1 M_2}{M_1 + M_2} + \frac{M_2 M_3}{M_2 + M_3}\right]$
4.	HeM	$\underline{y}^{-}_{n} + \frac{h}{6}\left[M_1 + 2M_2 + M_3 + \sqrt{\lvert M_1 M_2\rvert} + \sqrt{\lvert M_2 M_3\rvert}\right]$
5.	RM	$\underline{y}^{-}_{n} + \frac{h}{2}\left[\sqrt{\frac{M_1^2 + M_2^2}{2}} + \sqrt{\frac{M_2^2 + M_3^2}{2}}\right]$
6.	CeM	$\underline{y}^{-}_{n} + \frac{1}{3}h\left[\frac{M_1^2 + M_1 M_2 + M_2^2}{M_1 + M_2} + \frac{M_2^2 + M_2 M_3 + M_3^2}{M_2 + M_3}\right]$
7.	CoM	$\underline{y}^{-}_{n} + \frac{h}{2}\left[\frac{M_1^2 + M_2^2}{M_1 + M_2} + \frac{M_2^2 + M_3^2}{M_2 + M_3}\right]$

Table 4.4

S.No	**Means**	$\overline{y}^{+}_{n+1}$
1.	AM	$\overline{y}^{-}_{n} + \frac{h}{4}\left[N_1 + 2N_2 + N_3\right]$
2.	GM	$\overline{y}^{-}_{n} + \frac{h}{2}\left[\sqrt{\lvert N_1 N_2\rvert} + \sqrt{\lvert N_2 N_3\rvert}\right]$

3.	HaM	$\bar{y}_n + h\left[\frac{N_1 N_2}{N_1+N_2}+\frac{N_2 N_3}{N_2+N_3}\right]$
4.	HeM	$\bar{y}_n + \frac{h}{6}\left[N_1+2N_2+N_3+\sqrt{\lvert N_1 N_2\rvert}+\sqrt{\lvert N_2 N_3\rvert}\right]$
5.	RM	$\bar{y}_n + \frac{h}{2}\left[\sqrt{\frac{N_1^2+N_2^2}{2}}+\sqrt{\frac{N_2^2+N_3^2}{2}}\right]$
6.	CeM	$\bar{y}_n + \frac{1}{3}h\left[\frac{N_1^2+N_1N_2+N_2^2}{N_1+N_2}+\frac{N_2^2+N_2N_3+N_3^2}{N_2+N_3}\right]$
7.	CoM	$\underline{y}_n^- + \frac{h}{2}\left[\frac{N_1^2+N_2^2}{N_1+N_2}+\frac{N_2^2+N_3^2}{N_2+N_3}\right]$

5. Convergence of Intuitionistic Fuzzy Runge-Kutta Methods:

The solution is obtained by grid points at

$$a = t_0 \le t_1 \le \cdots\cdots \le t_N = b \text{ and } h = \frac{b-a}{N} = t_{n+1} - t_n \quad (5.1)$$

We define

$$F[t_n, y(t_n;a)] = \sum_{i=1}^{s} b_i K_i(t_n, y(t_n;a)) \quad G[t_n, y(t_n;a)] = \sum_{i=1}^{s} b_i L_i(, y(t_n;a))$$

$$H[t_n, y(t_n;\beta)] = \sum_{i=1}^{s} b_i M_i(t_n, y(t_n;\beta)) \; I[t_n, y(t_n;\beta)] = \sum_{i=1}^{s} b_i N_i(t_n, y(t_n;\beta)) \quad (5.2)$$

The exact and approximate solutions at t_n, $0 \le n \le N$ are denoted respectively by

$[Y(t_n)]_{\alpha,\beta} = [\underline{Y}^+(t_n;\alpha), \overline{Y}^+(t_n;\alpha), \underline{Y}^-(t_n;\beta), \overline{Y}^-(t_n;\beta)]$ and

$[y(t_n)]_{\alpha,\beta} = [\underline{y}^+(t_n;\alpha), \overline{y}^+(t_n;\alpha), \underline{y}^-(t_n;\beta), \overline{y}^-(t_n;\beta)]$.

We have

$$\underline{Y}^+(t_{n+1};\alpha) \approx \underline{Y}^+(t_n;\alpha) + hF[t_n, \underline{Y}^+(t_n;\alpha), \overline{Y}^+(t_n;\alpha)],$$
$$\overline{Y}^+(t_{n+1};\alpha) \approx \overline{Y}^+(t_n;\alpha) + hG[t_n, \underline{Y}^+(t_n;\alpha), \overline{Y}^+(t_n;\alpha)],$$
$$\underline{Y}^-(t_{n+1};\beta) \approx \underline{Y}^-(t_n;\beta) + hH[t_n, \underline{Y}^-(t_n;\beta), \overline{Y}^-(t_n;\beta)],$$
$$\overline{Y}^-(t_{n+1};\beta) \approx \overline{Y}^-(t_n;\beta) + hI[t_n, \underline{Y}^-(t_n;\beta), \overline{Y}^-(t_n;\beta)].$$

and

$$\underline{y}^+(t_{n+1};\alpha) = \underline{y}^+(t_n;\alpha) + hF[t_n, \underline{y}^+(t_n;\alpha), \overline{y}^+(t_n;\alpha)],$$
$$\overline{Y}^+(t_{n+1};\alpha) = \overline{y}^+(t_n;\alpha) + hG[t_n, \underline{y}^+(t_n;\alpha), \overline{y}^+(t_n;\alpha)],$$
$$\underline{Y}^-(t_{n+1};\beta) = \underline{y}^-(t_n;\beta) + hH[t_n, \underline{y}^-(t_n;\beta), \overline{y}^-(t_n;\beta)],$$
$$\overline{Y}^-(t_{n+1};\beta) = \overline{y}^-(t_n;\beta) + hI[t_n, \underline{y}^-(t_n;\beta), \overline{y}^-(t_n;\beta)].$$

We need the following lemmas to show the convergence of these approximates, that is, $\underline{y}^+(t_n;\alpha)$, $\overline{y}^+(t_n;\alpha)$, $\underline{y}^-(t_n;\alpha)$, and $\overline{y}^-(t_n;\alpha)$ converges to $\underline{Y}^+(t_n;\alpha)$, $\overline{Y}^+(t_n;\alpha)$, $\underline{Y}^-(t_n;\alpha)$, and $\overline{Y}^-(t_n;\alpha)$ respectively whenever $h \to 0$.

Lemma 5.1:

Let the sequence of numbers $\{W_n^+\}_{n=0}^{N}, \{W_n^-\}_{n=0}^{N}$ satisfy

$$|W_{n+1}| \le A|W_n| + B \quad 0 \le n \le N-1$$

for some given positive constants A and B. Then $|W_n| \le A^n |W_0| + B\dfrac{A^n - 1}{A-1}$.

Lemma 5.2:

Let the sequence of numbers $\{W_n\}_{n=0}^{N}$, $\{V_n\}_{n=0}^{N}$ satisfy

$$|W_{n+1}| \le |W_n| + A.\max\{|W_n||V_n|\} + B,$$

$$|V_{n+1}| \le |V_n| + A.\max\{|W_n||V_n|\} + B,$$

for some given positive constants A and B and denote $U_n = |W_n| + |V_n|, \quad 0 \le n \le N$.

Then $U_n \le \overline{A}^n U_0 + \overline{B}\dfrac{\overline{A}^n - 1}{\overline{A} - 1}, \quad 0 \le n \le N$ where $\overline{A} = 1 + 2A$ and $\overline{B} = 2B$..

Let F (t, u, v) and G (t, u, v) are obtained by substituting $[y(t)]_r = [u, v]$ in (5.2).
The domain where F and G are defined is therefore

$$K = \{(t,u,v)/0 \le t \le T, \quad -\infty < v < \infty, \ -\infty < u \le v\}.$$

Theorem 5.1:

Let F (t, u, v), G (t, u, v), H (t, u, v) and I(t, u, v) belong to $c^P(K)$ and let the partial derivatives of F and G be bounded over K. Then for arbitrary fixed $\alpha, \beta : 0 \le \alpha, \beta \le 1$, the approximate solution of (5.1), $\left[\underline{y}^+(t_n;\alpha),\ \overline{y}^+(t_n;\alpha), \underline{y}^-(t_n;\beta),\ \overline{y}^-(t_n;\beta)\right]$ converges to the exact solution $\left[\underline{Y}^+(t_n;\alpha),\ \overline{Y}^+(t_n;\alpha), \underline{Y}^-(t_n;\beta),\ \overline{Y}^-(t_n;\beta)\right]$.

Proof:

Byusing Taylor's theorem

$$\underline{Y}^+(t_{n+1};\alpha) = \underline{Y}^+(t_n;\alpha) + hF(t_n, \underline{Y}^+(t_n;\alpha), \overline{Y}^+(t_n;\alpha)) + \frac{h^{p+1}c_i^{\,p}}{(p+1)!}\underline{Y}^{+(p+1)}(\xi_{n,1})$$

$$\overline{Y}^+(t_{n+1};\alpha) = \overline{Y}^+(t_n;\alpha) + hG(t_n, \underline{Y}^+(t_n;\alpha), \overline{Y}^+(t_n;\alpha)) + \frac{h^{p+1}c_i^{\,p}}{(p+1)!}\overline{Y}^{+(p+1)}(\xi_{n,2})$$

$$\underline{Y}^-(t_{n+1};\beta) = \underline{Y}^-(t_n;\beta) + hH(t_n, \underline{Y}^-(t_n;\beta), \overline{Y}^-(t_n;\beta)) + \frac{h^{p+1}c_i^{\,p}}{(p+1)!}\underline{Y}^{-(p+1)}(\xi_{n,3})$$

$$\overline{Y}^-(t_{n+1};\beta) = \overline{Y}^-(t_n;\beta) + hI(t_n, \underline{Y}^-(t_n;\beta), \overline{Y}^-(t_n;\beta)) + \frac{h^{p+1}c_i^{\,p}}{(p+1)!}\overline{Y}^{-(p+1)}(\xi_{n,4})$$

where $\xi_{n,1}, \xi_{n,2}, \xi_{n,3}, \xi_{n,4} \in (t_n, t_{n+1})$

Now if we denote

$$W_n^+ = \underline{Y}^+(t_n,\alpha) - \underline{y}^+(t_n,\alpha),\ V_n^+ = \overline{Y}^+(t_n,\alpha) - \overline{y}^+(t_n,\alpha)$$

$$W_n^- = \underline{Y}^-(t_n,\beta) - \underline{y}^-(t_n,\beta) \text{ and } V_n^- = \overline{Y}^-(t_n,\beta) - \overline{y}^-(t_n,\beta)$$

then the above two expressions converted to
Hence we can write

$$|W_{n+1}^+| \le |W_n^+| + 2Lh\max\left(|W_n^+|, |V_n^+|\right) + \frac{h^{p+1}}{(p+1)!}M_1$$

$$|V_{n+1}^+| \le |V_n^+| + 2Lh\max\left(|W_n^+|, |V_n^+|\right) + \frac{h^{p+1}}{(p+1)!}M_1$$

$$|W_{n+1}^-| \le |W_n^-| + 2Lh\max\left(|W_n^-|, |V_n^-|\right) + \frac{h^{p+1}}{(p+1)!}M_2$$

$$|V_{n+1}^-| \le |V_n^-| + 2Lh\max\left(|W_n^-|, |V_n^-|\right) + \frac{h^{p+1}}{(p+1)!}M_2$$

Where

$$M_1 = \max\left\{\max\left|\underline{Y}^{+(p+1)}(t,\alpha)\right|, \max\left|\overline{Y^{+}}^{(p+1)}(t,\alpha)\right|\right\}$$

$$M_2 = \max\left\{\max\left|\underline{Y}^{-(p+1)}(t,\beta)\right|, \max\left|\overline{Y^{-}}^{(p+1)}(t,\beta)\right|\right\} \quad for\, t \in [0,\mathrm{T}],$$

and L > 0 is a bound from the partial derivative of F and G.
Therefore we can write,

$$|W_n^+| \le (1+4\mathrm{Lh})^n |U_0^+| + \left(\frac{2h^{p+1}}{(p+1)!}M_1\right)\frac{(1+4\mathrm{Lh})^n - 1}{4Lh},$$

$$|V_n^+| \le (1+4\mathrm{Lh})^n |U_0^+| + \left(\frac{2h^{p+1}}{(p+1)!}M_1\right)\frac{(1+4\mathrm{Lh})^n - 1}{4Lh}$$

$$|W_n^-| \le (1+4\mathrm{Lh})^n |U_0^-| + \left(\frac{2h^{p+1}}{(p+1)!}M_2\right)\frac{(1+4\mathrm{Lh})^n - 1}{4Lh},$$

$$|V_n^-| \le (1+4\mathrm{Lh})^n |U_0^-| + \left(\frac{2h^{p+1}}{(p+1)!}M_2\right)\frac{(1+4\mathrm{Lh})^n - 1}{4Lh}$$

Where $|U_0^+| = |W_0^+| + |V_0^+|$ and $|U_0^-| = |W_0^-| + |V_0^-|$
In particular

$$|W_n^+| \le (1+4\mathrm{Lh})^N |U_0^+| + \left(\frac{2h^{p+1}}{(p+1)!}M\right)\frac{(1+4\mathrm{Lh})^{\frac{T}{h}} - 1}{4Lh},$$

$$|V_n^+| \le (1+4\mathrm{Lh})^N |U_0^+| + \left(\frac{2h^{p+1}}{(p+1)!}M\right)\frac{(1+4\mathrm{Lh})^{\frac{T}{h}} - 1}{4Lh}$$

$$|W_n^-| \le (1+4\mathrm{Lh})^N |U_0^-| + \left(\frac{2h^{p+1}}{(p+1)!}M\right)\frac{(1+4\mathrm{Lh})^{\frac{T}{h}} - 1}{4Lh},$$

$$|V_n^-| \le (1+4\mathrm{Lh})^N |U_0^-| + \left(\frac{2h^{p+1}}{(p+1)!}M\right)\frac{(1+4\mathrm{Lh})^{\frac{T}{h}} - 1}{4Lh}$$

Where

$$M = \max\{M_1, M_2\} \quad for\, t \in [0,\mathrm{T}],$$

Since $W_0^+ = V_0^+ = W_0^- = V_0^- = 0,$ we have

$$|W_n^+| \le M\frac{(e^{4\mathrm{LT}}-1)}{2L(p+1)!}h^p, \quad |V_n^+| \le M\frac{(e^{4\mathrm{LT}}-1)}{2L(p+1)!}h^p,$$

$$|W_n^-| \le M\frac{(e^{4\mathrm{LT}}-1)}{2L(p+1)!}h^p, \quad |V_n^-| \le M\frac{(e^{4\mathrm{LT}}-1)}{2L(p+1)!}h^p$$

Thus if $h \to 0$ we get $W_n^+ \to 0,\ V_n^+ \to 0,\ W_n^- \to 0$ and $V_n^- \to 0$ which completes the proof.

6. Numerical Examples:

Example 6.1: Oil Production Model

In the world rate of increase oil production y in million metric tons per year was assumed to be proportional to y itself. Then what is the amount of oil after five years initially (1061, 1091, 1131; 1066, 1091, 1111) million metric ton oil (the constant of proportionality is 0.084).

Solution:

$\frac{dy}{dt} = ky$ with k=0.084 and $y_0 = (1061,\ 1091,\ 1131;\ 1066,\ 1091,\ 1111)$ million.

The exact solution is given by

$$\underline{y}^{+}(t;\alpha)=(1061+30\alpha)e^{kt} \qquad \overline{y}^{+}(t;\alpha)=(1131-40\alpha)e^{kt}$$

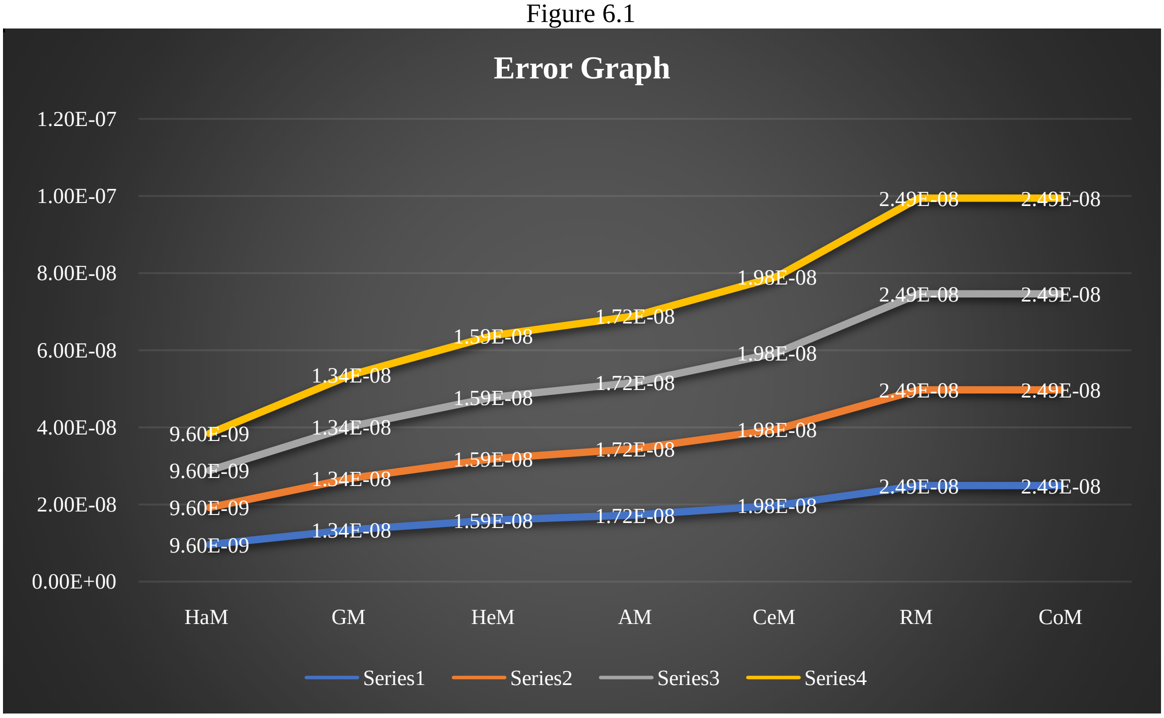
$$\underline{y}^{-}(t;\beta)=(1091-25\alpha)e^{kt} \qquad \overline{y}^{-}(t;\beta)=(1091+20\alpha)e^{kt}$$

The Error results for example 6.1 at t =1 and $(\alpha,\beta)=1$ are shown in the Table 6.1. The error graph are shown Figure 6.1.

Table 6.1

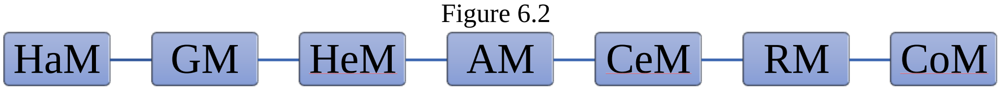
Means	**Absolute Error for IFRK3 at t=1**			
	$\underline{y}^{+}(t;\alpha)$	$\overline{y}^{+}(t;\alpha)$	$\underline{y}^{-}(t;\beta)$	$\overline{y}^{-}(t;\beta)$
HaM	9.598e-09	9.598e-09	9.598e-09	9.598e-09
GM	1.340e-08	1.340e-08	1.340e-08	1.340e-08
HeM	1.594e-08	1.594e-08	1.594e-08	1.594e-08
AM	1.721e-08	1.721e-08	1.721e-08	1.721e-08
CeM	1.976e-08	1.976e-08	1.976e-08	1.976e-08
RM	2.485e-08	2.485e-08	2.485e-08	2.485e-08
CoM	2.485e-08	2.485e-08	2.485e-08	2.485e-08

Figure 6.1



By analyzing the absolute errors from the Tables, the RK3 methods based on variety of means could be sorted under different levels subject to minimum error. The pictorial representation is given in Fig. 6. 1.

Figure 6.2



Conclusion:

In this paper, the third order Runge Kutta methods based on AM, HeM, CeM, RM, GM, HaM and CoM have been presented to solve the intuitionistic fuzzy initial value problems. The efficiency of the method has been illustrated through the example of linear problem of Triangular type. The numerical results obtained are agreed well with the corresponding exact solutions. This shows that the third order RK method based on the above variety of means are much applicable to solve intuitionistic fuzzy initial value problems.

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