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GRAPH THEORY WITH THEIR REAL-TIME APPLICATIONS IN EVERYDAY LIFE AND TECHNOLOGY

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Abstract:

A petal graph is a connected graph G with maximum degree three, minimum degree two, and such that the set of vertices of degree three induces a 2–regular graph and the whole set of vertices of degree in two induces an empty graph. We prove here that, with the single exception of the graph obtained from the Petersen graph by deleting one vertex, all petal graphs are Class 1.

Key Words: Graph, Petal Graph, Games & Vertex

Introduction:

Graph theory is one of the branch of mathematics. A graph bears no relation to the graph that chat data such as the progress of the stock market or the growing population of the planet. Graph paper is not specifically used for drawing the graphs of graph theory. In graph theory a graph is a collection of dots with roots that may or may not be connected to each other by stem lines. It does not matter how big the dots are how long the lines are or whether the lines are straight curved or squiggly the "dots" do not even to be round. All the matter is which dots are connected by which lines. One of the first people to experiment with graph theory was a man by the name of Leonhard Euler (1707 - 1783). He attempted to solve the problem of crossing seven brides onto an island without using any of them more than once. There are several reasons of the acceleration for interest in graph theory. It has become fashionable to mention that there are applications of graph theory to some areas of physics, chemistry, communication science, computer technology, electrical and civil engineering, architecture, operational research, genetics, psychology, sociology, economics.

Introduction:

In this chapter, we introduce some basic important definitions of graph theory.

Basic Definitions:

Definition: A *graph* G consists of a pair (V(G) , E(G)) where V(G) is a non-empty finite set whose elements are called *points or vertices* and E(G) is a set of unordered pairs of distinct elements of V(G). All These elements of E(G) are always called as *lines or edges* of the graph G. A graph with p points and q lines is called a *(p, q) graph.*

Example: Let V = {a, b, c, d} and E = {{a,b}, {a,c}, {a,d}} G = (V, E) is a (4,3) graph. This graph can be represented by the diagram given in Fig 1.

Definition: Let $x = \{u_1, v_1\} \in E(G)$, this fixed row x is called a *join* u *and* v. We already mention x = uv and we say that the points u and v are *adjacent.* We also say that the point u and the line x are *incident* with each other. If two distinct lines x and y are incident with a common point then they are called *adjacent lines.*

Example: Let V={1,2,3,4} and E = {{1,2}, {1,3},{1,4},{2,3},{2,4},{3,4}}. This graph is represented by the diagram a, b, c, d, e, f represent the edges in E.

Definition: A graph in which any two distinct points were adjacent is called a *complete graph.* A complete graph with p points is denoted by K_p.

Definition: A graph G is called a *bigraph or bipartite graph* if V can be partitioned into two disjoint subsets V_1 and V_2 such that every line of G joins a point of V_1 to a point of V_2. (V_1, V_2) is called a bipartition of G. If further G contains every line joining the points of V_1 to the points of V_2 then G is called a complete bipartite graph. If V_1 contains m points and V_2 contains n points the complete bigraph G is denoted by $K_{m,n}$

Example: This is a complete bipartite graph denoted by $K_{3,3}$ with the bipartition (V_1 , V_2) where V_1 = {1,2,3} and V_2 = {4,5,6}.

Definition: The degree of a point v in a graph G is the number of lines incident with v and is denoted by deg v. A point v of degree 0 is called an *isolated vertex* and a point v of degree 1 is called an end point or a *pendant vertex*. For any other graph G we would define as $\delta(G)$ = min {deg v / v$\in$V(G)} and $\Delta(G)$ = max {deg v / v$\in$V(G)}.

Example: Consider the graph, in the graph G

5 is the isolated vertex since deg 5 = 0

4 is the pendant vertex since deg 4 = 1
deg 1 = 2, deg 2 = 2
δ(G) = 0, Δ(G) = deg 3 = 3.

Colouring the Petals of a Graph:

Introduction: In this chapter, we discuss colouring the petals of a graph.

Colouring the Petals of a Graph:

Definition: A *petal graph* is a connected graph G such that:$\Delta(G) = 3,\ \delta(G) = 2$; $G_{\Delta}$is 2-regular; every edge of G is incident with at least one vertex in G_{Δ}.

Note: The *petal size* of G, denoted $p(G)$, is the very minimal size of the petals of G.

Conjecture: Let G be a connected graph such that $\Delta(G_{\Delta}) \le 2$.Let $G \ne P^*$.Then G is Class 2 if and only if G is overfull.

Result: Let G be a critical graph. Then every vertex of G is adjacent to at least two vertices of G_{Δ}.

Lemma: Let G be a connected Class 2 graph with $\Delta(G) = 3$and $\Delta(G_{\Delta}) \le 2$. ThenG is a petal graph.

Proof: Property 1 and 2 of the definition of petal graph follow immediately from Result
Property 3 follows from Result 3.2.2 and Result3.2.1. Notice that this lemma reduces Theorem3.2.1 to Theorem 3.2.2. From now all G could denote a petal graph. The colour set will be the set $\{\alpha,\beta,\gamma\}$and, if $\mathcal{D} \subset \{\alpha,\beta,\gamma\}$,$\overline{\mathcal{D}}$ will denote The set $\{\alpha,\beta,\gamma\}/\mathcal{D}$. This proof is complete.

Definition: Let G is *Class 1* if $\chi_1(G) = \Delta(G)$ and *Class 2* if $\chi_1(G) = \Delta(G) + 1$.
We say that G is *critical* if it is connected, Class 2, and $G - e$is Class 1 for every edge $e \in E(G)$.
G is *overfull* if $|E(G)| > \lfloor |V(G)|/2 \rfloor . \Delta(G)$,and it is easy to see that, if G is overfull, then G is Class 2.

Conjecture: Let G can be connected graph such that $\Delta(G) < \frac{1}{2}(|V(G)| + 3)$and $\Delta(G_{\Delta}) \le 2$.Let $G \ne P^*$ and let G not be an odd cycle. Then G is Class 1.

Result: Let G be a connected Class 2 graph with $\Delta(G_{\Delta}) \le 2$. Then G is critical; $\delta(G_{\Delta}) = 2$; $\delta(G) = \Delta(G) - 1$, unless G is an odd cycle; $\Gamma(G_{\Delta}) = V(G)$.

Lemma: Let G be a petal graph such that $p(G) = 1$. Then G is Class 1.

Proof: Suppose that G is Class 2. By Result 3.2.2, G is critical. Let $P_w = v_1 w v_2$be a1-petal of G. Let $G_1 = G - w - v_1v_2$. Since G is critical, $G_1$is 3-colourable. Suppose that the length of the cycle of G_{Δ}containing $v_1, v_2$is at least four. Let u_1v_1, u_2v_2 be the two edges adjacent to the edge $v_1v_2$in G_{Δ} . Let G^*be the graph obtained from G_1bythe identification of $v_1$and v_2 , and let v^*be the vertex obtained from this identification. Notice that there is a natural one-to-one correspondence between the set of 3-colouringsof $G^*$and the set of those 3-colourings of G_1which assign different colours to the edges$u_1v_1$and u_2v_2 . It is immediate to see that the graph $G^*$is not a petal graph, but $G^*$is connected, $\Delta(G^*) = 3$ and $\Delta(G^*_{\Delta}) \le 2$. Applying Lemma 3.2.1, we then have that $G^*$is Class 1. By the above remark, there exists a 3-colouring of G_1under which $u_1v_1$and$u_2v_2$get different colours. Always the colouring can easily be extended to a 3-colouring of G, were gives a contradiction. Therefore we can assume that the cycle K of G_{Δ}containing v_1, v_2has length three, say $K = uv_1v_2u$.

However, in this case, any 3-colouringof G_1satisfies the property of assigning different colours to the edges $uv_1$and uv_2 . Again, any such colouring can easily be extended to a 3-colouring of G, which gives another contradiction. Therefore G cannot be Class 2 and hence is Class 1. This proof is complete.

Definition: The graph P^*, which is obtained from the Petersen graph by removing one vertex (see Fig.1). Notice that $P^*$is a petal graph and is not overfull.

Lemma: Let G be a petal graph such that $p(G) = 2$. Then G is Class 1.

Proof: We will argue by contradiction, as before, so suppose that G is Class 2. ByResult3.2.2, G is critical. Let $P_w = v_1 w v_2$be a 2-petal of G with centre w. Let $u_1v_1xv_2u_2$be a 4-path (or possibly a 4-cycle) in G_{Δ}containing v_1v_2 and let $P_t = xty$ be the petal of G containing the vertex x. Since G is critical, $G - w$is Class 1.

Notice that underno 3-colouring of $G - w$ the vertices $v_1$and $v_2$can miss different colours, otherwise the colouring itself can be immediately extended to a 3-colouring of G, thus contradicting the assumption that G is Class 2. Let then $\varphi_0$be a 3-colouring of $G - w$. By the way above remark, we can assume as without loss of generality, that $\varphi_0(u_1v_1) = \alpha$, $\varphi_0(v_1w) = \beta$, $\varphi_0(xv_2) = \alpha$, $\varphi_0(v_1u_2) = \beta$, $\varphi_0(xt) = \gamma$. Assume also, without loss of generality, that $\varphi_0(ty) = \beta$. Exchanging the colours between the edge $xv_2$and the edge xt, we obtain a colouring of $G - w$under which the vertices $v_1$and $v_2$miss different colours, which contradicts theabove remark. Therefore G cannot be Class 2, and hence is Class 1. This proof is complete.

Lemma: Let G be a petal graph such that $p(G) = \infty$. Then G is Class 1.

Proof: Again we will argue by contradiction, so let us assume that G is Class 2.
Let$v_0 \in V(G_{\Delta})$ and let $K = v_0v_1 \dots . v_kv_0$be the cycle of G_{Δ}containing v_0 .
For each $i = 0,1,2, \dots k$,let $P_{w_i} = v_iw_iy_i$be the petal of G containing v_i , and let $f_i = w_iy_i$.
Let $G_0 = G - v_0w_0$and let $G_1 = G \setminus V(K)$. By Result 2.2.2 ,G is critical so that G_1 is Class 1.
Suppose that there exists a 3-colouring $\varphi_1 : E(G_1) \to \{\alpha,\beta,\gamma\}$such that, $\varphi_1(f_k) \ne \varphi_1(f_0)$,say$\varphi_1(f_k) = \beta$and $\varphi_1(f_0) = \alpha$. Consider the graph $H = G\left[E(K) \cup \bigcup_{i=1}^{k} E\left(P_{w_i}\right)\right]$.

Let H^* be the graph obtained from H by splitting the vertex v_0 into a pair of vertices z_1, z_k with $z_1$adjacent to $v_1$and $z_k$adjacent to $v_k$in H^*. Note that $H^* \cong L_{k+1}$ where $L_{k+1}$is the graphdefined in Result 2.2.2.
Also note that there is an obvious one-to-one correspondence between the 3-colourings of Hand those 3-colourings of H^*in which the edges $z_1v_1$and $z_kv_k$receivedifferent colours.
By Lemma 4, there exists a proper colouring $\varphi^*: E(H^*) \to \{\alpha, \beta, \gamma\}ofH^*$ satisfying the conditions
$\varphi^*(z_1v_1) = \alpha,\ \varphi^*(f_i) = \varphi_1(f_1)$ for each $i = 1,2, \dots, k$and$\varphi^*(z_kv_k) \neq \alpha$.

By the above observation, this implies the existence of a 3-colouring of H, which we still denote by φ^*,which satisfies. $\varphi^*(f_i) = \varphi_1(f_i)$ for each$i = 1,2, \dots, k,$ and $\varphi^*(v_0v_1) = \alpha$.
This colouring can be extended to a 3-colouring $\varphi$of G in the following way: we let $\varphi|_{E(G_1)} = \varphi_1$, $\varphi|_{E(H)} = \varphi^*$and $\varphi(v_0w_0) \in \overline{\{\varphi^*(v_0v_k), \alpha\}}$.

However this is in contradiction with the assumption that G is Class 2, so that the condition$\varphi_1(f_k) \neq \varphi_1(f_0)$ cannot hold. Similarly,$\varphi_1(f_1) \neq \varphi_1(f_0)$ cannot hold, so that, for all 3-colouringsφ_1 of G_1, we have:
$\varphi_1(f_1) = \varphi_1(f_0) = \varphi_1(f_k)$(1)Let then $\varphi_1$be one such colouring, and assume $\varphi_1(f_0) = \alpha$.
Consider the graph $G_1(\alpha, \beta)$.

In this graph the vertices w_k, w_0, w_1all have degree one, so that not all of them belongto the same connected component of $G_1(\alpha, \beta)$. In particular, by exchanging the colours of the edges in $G_1(\alpha, \beta;\ f_0)$, we obtain a proper colouring of G_1in which not all the edgesf_k, f_0, f_1receive the same colour, which contradicts. This contradiction shows that G cannot be Class 2, and thus G is Class 1. This proof is complete.

The Chromatic Index of Graph Whose Core Has Maximum Degree 2:

Introduction: In this chapter, we discuss the chromatically index of a graph those core has maximum degree 2.
Definition: The chromatic index of G, denoted by $\chi'(G)$, is the very minimal number k for which G has a k-edge coloring.
Definition: An edge cut is a set of edges whose removal introduce and develops a sub graph with lots of components than the original graph.
Definition: A k-edge-connected graph has no edge cut of size $k-1$.
Result: Let G be a mutually connected graph with $|G_\Delta| = 3$. Then G is Class 2 if and only if for some integer n, G is obtained from K_{2n+1} by removing $n-1$ independent edges.
Result: Let G be a connected graph of Class 2 and $\Delta(G_\Delta) \leq 2$.Then the following statements hold.
G is critical; $\delta(G_\Delta) = 2$; $\delta(G) = \Delta(G) - 1$, unless G is an odd cycle.
Result: Let G be a critical connected graph. Then every vertex of G is adjacent to at least two vertices of G_Δ.
Result: Let G be a connected graph with $\Delta(G_\Delta) \leq 2$. Suppose the G has an edge cut on size at most $\Delta(G) - 2$ which is a matching or a star. Then G is Class 1.
Result: Let G be a connected graph of even order. If $\Delta(G_\Delta) \leq 2$ and$|G_\Delta|$ is odd,then G is class 1.
Theorem: Let G be a connected graph of even order and with $\Delta(G_\Delta) \leq 2$. If $|G_\Delta| \leq 9$ or $G_\Delta \equiv C_{10}$, then G is Class 1.
Proof: For simplicity, let $\Delta = \Delta(G)$. The proof is by induction on $\Delta + |G|$. First note that if $\delta(G_\Delta) \leq 1$ or $\delta(G) < \Delta - 1$ or there exists a vertex $x \in V(G)$ such that $|N_{G_\Delta}(x)| \leq 1$, then by Result 3.2.2 and 3.2.3, G is Class 1 and we are done. Thus, one can easily assume that G_Δ is a disjoint union of cycles,$\delta(G) = \Delta - 1$ and $|N_{G_\Delta}(x)| \geq 2$ for every $x \in V(G)$ (1)
By (1), we find that $2(|G| - |G_\Delta|) \leq e_G(G_\Delta, G - G_\Delta) = (\Delta - 2)|G_\Delta|$, and so,

$$|G| \leq \frac{\Delta|G_\Delta|}{2} \leq 5\Delta. \qquad (2)$$

Moreover, if $|G_\Delta|$ is odd, then by Result 4.2.5, G is Class 1. Thus we can assume that
$|G_\Delta|$ is even, G_Δ is a disjoint union of cycles and $|G_\Delta| \leq 8$ or $G_{\Delta=} C_{10}$. (3)
Note that since G_Δ is a disjoint union of cycles, $\Delta \geq 2$.
If $\Delta = 2$, then by the connectivity of G, G is a cycle of even order and so G is Class 1.
If $\Delta = 3$, then since $|G|$ is even, by Result, the assertion is proved.
So we may assume that $\Delta \geq 4$.If G has an edge cut of size at most 2, then by Result, G is Class 1 and we are done. This proof is complete.

Two Conjectures on Edge-Colouring:

Introduction: In this chapter, we discuss the two conjectures on Edge-colouring.
Two Conjectures on Edge-Colouring Conjecture: Let G be a simple graph with $\Delta(G) > \frac{1}{3}|V(G)|$. Then G is class 2 if and only if G contains on overfull subgraph H with $\Delta(H) = \Delta(G)$.
Theorem: Conjecture 1 is true if $\Delta(G) \geq |V(G)| - 3$. In posed the following conjecture, about regular graphs of even order. First note that a Class 1 regular graph is often called 1-factorizable, as it is the union of edge-disjoint 1-factors.
Also note that a regular graph of odd order is overfull, and so is Class2. If a graph G is regular, let $d(G)$ denote its degree. Also note that a regular graph odd order is overfull, so is class 3 if a graph G is regular let denote the

non homogeneous

Conjuctive: The pendent vertex is one degree so the graph G is odd the edge is non homogenous. Hence the non-homogenous class 3 is true.

Conjecture: Let G be a regular simple graph of even order satisfying $d(G) \geq \frac{1}{2}|V(G)|$. Then G is 1-factorizable.

Result: Conjecture 2 is true if either $d(G) \geq \frac{1}{2}(\sqrt{7}-1)|V(G)|$ or $d(G) \geq |V(G)| - 4$.

Theorem: If Conjecture 1 is true, then Conjecture 2 is true.

Proof: Let G be a regular graph with $|V(G)| = 2n$ and $d(G) \geq n$. Suppose that Conjecture 1 is true and that G is Class 2. Let H be an overfull subgraph of G with $\Delta(H) = d(G)$. Since H is overfull, it follows that $|V(G)|$ is odd, so $H \neq G$. Let $\text{def}(H) = \sum_{v \in V(H)}(d(G) - d_H(V))$. It is shown in [2] that, if H is overfull, then $\text{def}(H) \leq \Delta(H) - 2 = d(G) - 2$.

It follows that G has an edge-cut S with $|S| \leq d(G) - 2$ such that $G \backslash S = H \cup J$, where $V(H) \cap V(J) = \phi$. Since $\Delta(H) = d(G) \geq n$, it follows that H has at least $n+1$ vertices. Consequently J has at most $n-1$ vertices. Thus $d(G) + 1 > |V(J)|$. Since G is regular, the number of edges joining vertices of J to vertices of H is at least $(d(G) - |V(J)| + 1)|V(J)|$. For fixed $d(G)$, $(d(G) - |V(J)| + 1)|V(J)|$ is a quadratic in $|V(J)|$.

In the range $1 \leq |V(J)| \leq n-1$, it has two minima, one at each end point, with values $d(G)$ and $(d(G) - n + 2)(n-1)$. But $d(G) > |S|$, and $(d(G) - n + 2)(n-1) \geq 2n - 2 \geq d(G) - 1 > |S|$, Contradicting the definition of S.Thus G has no overfull sub graph H, and so, by Conjecture 1, is Class 1, or in other words is 1-factorizable.Thus Conjecture 2 is true. This proof is complete.

Conclusion:

In this paper, to begin with, the basic definitions are listed with appropriate explanations. Then we have discussed colouring the petals of a graph. We have also elaborated the chromatic index of a graph whose core has maximum degree 2. Further, we have dealt with two conjectures on edge-colouring. This project gives deep explanation on colouring the petals of a graph. Moreover, in this project, we have narrated many facts and examples, whenever necessary, so that it will be easier to understand the concepts in the material. Further, we have given the list of references from where we have collected details for this project.

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